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EDITED FOR THE MATHEMATICAL ASSOCIATION BY

R. L. GOODSTEIN

WITH THE ASSISTANCE OF

H. M. CUNDY K. M. SOWDEN

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EDITED BY

PROF. R. L. GOODSTEIN, UNIVERSITY COLLEGE OF LEICESTER

WITH THE ASSISTANCE OF

DR. H. MARTYN CUNDY, THE BEECHES, OBORNE ROAD, SHERBORNE, DORSET

MISS K. M. SOWDEN, NEWTON PARK COLLEGE, NEWTON ST. LOE, BATH

VOL. XL FEBRUARY, 1956 No. 331

CURVES OF PURSUIT

BY R. H. MACMILLAN

THE article by C. C. Puckette in a recent issue of the *Gazette* (Vol. 37, No. 322, pp. 256–260) prompts me to give some results I obtained when investigating the pursuit of a bomber by an attacking fighter.

Simple Pursuit

Consider the case in which the bomber takes no evasive action, but flies straight with constant speed, v_B. Let the fighter's speed be v_F, also constant, and let the ratio v_F/v_B be a. In the simple pursuit curve the fighter's velocity is directed towards the bomber at all times.

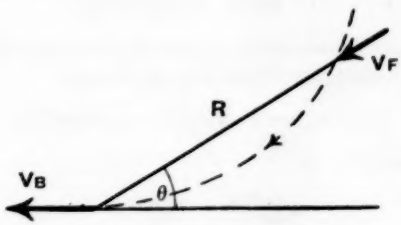

FIG. 1.

– – – – Path relative to bomber

If the range of the fighter is R when its angle off the bomber's tail is θ, as shown in Fig. 1, then we deduce as the equations of motion

$$\dot{R} = v_B \cos\theta - v_F$$

and

$$R\dot{\theta} = -v_B \sin\theta,$$

from which

$$\frac{dR}{R} = \frac{a - \cos\theta}{\sin\theta}\, d\theta.$$

Integrating this and adding a constant of integration R_0, we obtain

$$\log R = a \log \tan \tfrac{1}{2}\theta - \log \sin \theta + R_0$$

or

$$R = R_0 \tan^a \tfrac{1}{2}\theta \operatorname{cosec} \theta$$

as the polar equation of the fighter's path relative to the bomber. Clearly R_0 is the range of the fighter when it is on the bomber's beam, (i.e., $\theta = 90°$).

Lead Pursuit

This solution must be modified if the fighter attempts to fire at the bomber throughout the pursuit. In order that his bullets may strike, the fighter's gun direction must be pointed ahead of the bomber by the correct angle of "lead", which is equal to $R\dot{\theta}/v$, where v is the mean speed of the bullets relative to the fighter. As a first approximation v may be taken as constant at its mean value throughout the attack. The modified equations of motion are now

$$\dot{R} = v_B \cos \theta - v_F$$

and

$$R\dot{\theta}(1 - v_F/v) = -v_B \sin \theta$$

whose solution is found similarly to be

$$\frac{R}{R_0} = \left\{ \frac{\tan^a \frac{1}{2}\theta}{\sin \theta} \right\}^{(1-v_F/v)}.$$

Time to reach bomber

It will be observed that, for a fixed value of a, all pursuit curves relative to the bomber are similar, with centre of similitude at the bomber. It is therefore reasonable to make the guess that the time to reach the bomber from any position can be expressed in the form

$$t = Rf(\theta),$$

Differentiating,

$$-1 = \dot{R}f(\theta) + R(df(\theta)/d\theta)\dot{\theta},$$

the negative sign being introduced because t decreases as the fighter approaches the bomber. Substituting for $\dot{R}$ and $R\dot{\theta}$, we obtain

$$-1 = (v_B \cos \theta - v_F)f(\theta) - v_B \sin \theta f'(\theta),$$

which equation is satisfied by

$$f(\theta) = \frac{a + \cos \theta}{v_B(a^2 - 1)}$$

so

$$t = \frac{R(a + \cos \theta)}{v_B(a^2 - 1)}.$$

It is possible to draw contours joining all points from which the fighter will reach the bomber in the same time. For the normal case where $a > 1$, these are ellipses with the bomber at one focus. When $a = 1$, the simple pursuit curves degenerate into parabolas

$$R = R_0/(1 + \cos \theta)$$

with the bomber at the focus: the least distance the fighter can approach is thus $\frac{1}{2}R_0$, but the time to reach this position is always infinite. If $a < 1$, then

the point of closest approach is where $\cos\theta = a$, since $\dot{R}$ is then zero. The time contours are then hyperbolic.

Fighter acceleration

The limiting factor in making a pursuit attack may be the acceleration demanded of the fighter normal to its plane. Due to the curvature of its path, the fighter must have an acceleration $v_F\dot{\theta}$ (equal to ng, say) at all times, since it points along the line joining it to the bomber. We thus have

$$ng = v_F v_B \sin\theta / R.$$

Contours of constant acceleration are thus circles tangential at the origin. Acceleration is a maximum when $\ddot{\theta} = 0$, but since

$$R\dot{\theta} = -v_B \sin\theta$$

then
$$\dot{R}\dot{\theta} + R\ddot{\theta} = -v_B \cos\theta \,.\, \dot{\theta}$$

so at maximum acceleration

$$\dot{R} = -v_B \cos\theta\,;$$

but
$$\dot{R} = v_B \cos\theta - v_F,$$

from the equation of motion; whence $\cos\theta = \frac{1}{2}a$ where the acceleration is greatest. If $a > 2$, the acceleration continues to increase until the bomber is reached. The effect of " lead " is to increase the acceleration demanded.

Rate of roll

The maximum rate of roll necessary to obtain a required rate of change of acceleration may also be a limiting factor. Fig. 2 shows a head-on view of a fighter of mass M, at an angle of bank β, turning in a horizontal plane with acceleration ng; the forces acting upon it are shown.

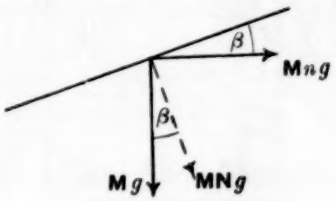

FIG. 2.

If there is no sideslip, the resultant force on the fighter must act normal to its plane, so that $\beta = \tan^{-1} n$, and the total load on the fighter and pilot is

$$Ng = \sqrt{(n^2+1)} \,.\, g.$$

Since $\tan\beta = n$, the rate of roll $d\beta/dt$ is related to the rate of change of n by

$$\frac{d\beta}{dt} = \frac{1}{n^2+1}\frac{dn}{dt},$$

from which expression it is possible to deduce the angle off the bomber at which the fighter's rate of roll must be a maximum.

Example

In Fig. 3 are plotted sets of pursuit curves with corresponding time and total acceleration contours for a fighter speed of 500 m.p.h. and a bomber speed of 300 m.p.h., i.e. $a = 1{\cdot}67$. The lines of maximum acceleration and maximum rate of roll are also marked. The pursuit curves drawn are those demanding maximum total accelerations of $1{\cdot}5g$, $2g$, $3g$, $4g$ and $5g$.

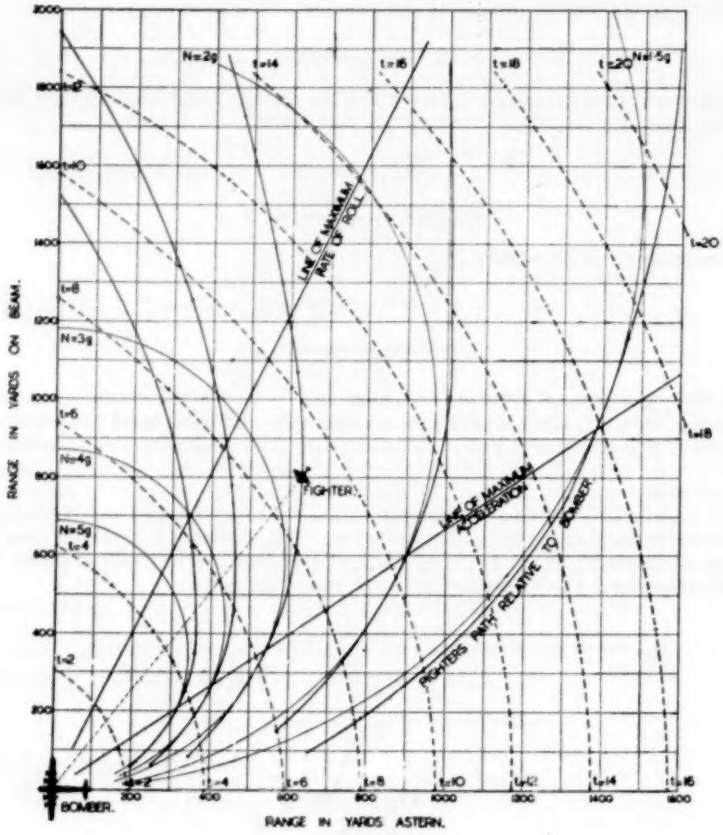

FIG. 3.

FIG. 3. Pursuit curves with contours of acceleration and time to reach the bomber. Bomber speed 300 m.p.h. ; fighter speed 500 m.p.h.

Acknowledgement

I am indebted to the Chief Scientist, Ministry of Supply, for permission to publish the results of work done when I was working in a Government Research establishment.

R. H. M.

RATIONAL ANALOGUES OF THE LOGARITHM FUNCTION

By J. Lambek and L. Moser

1. Introduction. If we abbreviate

$$h(n) = 1 + 1/2 + 1/3 + \ldots + 1/n, \quad \ldots\ldots\ldots\ldots\ldots\ldots (1.1)$$

a well known result becomes

$$h(n) = \log n + \gamma + O(1/n), \quad \ldots\ldots\ldots\ldots\ldots\ldots (1.2)$$

where γ is Euler's constant. $h(n)$ has properties approximating those of $\log n$, a fact of importance in number theory [see §6].

In this paper, we consider a sequence $l_r(x)$ $(r = 1, 2, \ldots)$ of rational functions of a positive rational variable x, which approach $\log x$ as $r \to \infty$. Some properties of these functions are analogous to properties of the natural logarithm, and go over into the latter, when the limit $r \to \infty$ is taken. Except for pointing out such limiting results, we shall avoid transcendental methods.

In what follows, the letters a, b, r, s, n will denote positive integers, x and y rationals.

2. The function $l_r(n)$.

Lemma 2.1. $a/(a+b) \leqslant h(a+b) - h(b) < a/b$.

Proof. The above difference, when expanded according to (1.1), contains a terms, each less than $1/b$ and none less than $1/(a+b)$.

Definition 2.2. $l_r(a) = h(ra) - h(r)$.

Lemma 2.3. $0 \leqslant l_{r+1}(a) - l_r(a) < 1/r - 1/(r+1)$.

Proof. The above difference is

$$h(ra+a) - h(r+1) - h(ra) + h(r) = h(ra+a) - h(ra) - 1/(r+1).$$

Now apply (2.1).

Lemma 2.4. $0 \leqslant l_s(a) - l_r(a) < 1/r - 1/s, \quad (r < s)$.

Proof. This follows by repeated application of (2.3).

Theorem 2.5. $0 \leqslant l_r(ab) - l_r(a) - l_r(b) < 1/r$.

Proof. The expression to be estimated is

$$h(rab) - h(ra) - h(rb) + h(r) = l_{ra}(b) - l_r(b).$$

Now apply (2.4) with $s = ra$.

Thus the function l_r defined on integers satisfies the functional equation of the logarithm, with an " error " of less than $1/r$. By induction on n we obtain the

Corollary 2.6. $0 \leqslant l_r(a^n) - n l_r(a) < (n-1)/r$.

Corresponding to $d \log x/dx = 1/x$ we have

Proposition 2.7. $1/(a+1) \leqslant l_r(a+1) - l_r(a) < 1/a$.

Proof. In view of (2.2), the above difference is $h(ra+a) - h(ra)$. The result follows from (2.1).

3. The function $l_r(x)$. We now define l_r for rational arguments.

Definition 3.1. If $x = a/b$ with $(a, b) = 1$, then $l_r(x) = h(ra) - h(rb)$.

Note that this reduces to (2.2) when $b = 1$.

LEMMA 3.2. If $(a, b)=1$, then $l_r(a/b)=l_r(a)-l_r(b)$.

Proof. This follows directly from (2.2) and (3.1).

LEMMA 3.3. $|\, l_r(a)-l_r(b)-l_r(a/b)\,|<1/r$.

Proof. Let $(a, b)=m$, $a=cm$, $b=dm$. Then the expression to be estimated becomes

$$h(rcm)-h(rdm)-h(rc)+h(rd)=l_{rc}(m)-l_{rd}(m).$$

Now apply (2.4).

THEOREM 3.4. $|\, l_r(xy)-l_r(x)-l_r(y)\,|<2/r$.

Proof. Let $x=a/b$, $y=c/d$, $(a, b)=(c, d)=1$.

In view of (3.3) it suffices to prove that

$$|\, l_r(ac)-l_r(bd)-l_r(a/b)-l_r(c/d)\,| \leqslant 1/r.$$

By (3.2) this is

$$(l_r(ac)-l_r(a)-l_r(c))-(l_r(bd)-l_r(b)-l_r(d)).$$

By (2.5) each of these two expressions lies between 0 and $1/r$, hence their difference lies between $-1/r$ and $1/r$.

4. The function $\gamma_r(n)$. Of interest in view of (1.2) is the following

Definition 4.1. $\gamma_r(s)=h(s)-l_r(s)$.

Proposition 4.2. $\gamma_r(s)=\gamma_s(r)$.

Proof. Both sides are equal to $h(s)+h(r)-h(rs)$.

Proposition 4.3. $0 \leqslant \gamma_r(a)-\gamma_s(a)<1/r-1/s \quad (r<s)$.

Proof. $\gamma_r(a)-\gamma_s(a)=l_s(a)-l_r(a)$ by (4.1).

Now apply (2.4).

5. *Conclusions.* In the above definitions and proofs all limiting processes have been avoided. From theorem 3.4 it is clear that $l_r(x)$ may be used effectively to replace $\log x$ in many situations. If we now invoke the Cauchy convergence criterion, we deduce directly from (2.4) that $\lim_{r\to\infty} l_r(a)$ exists. The existence of $\lim_{r\to\infty} l_r(x)$ then follows from (3.2). If we denote this limit by $l(x)$, theorem 3.4 leads to an identification of $l(x)$ with $\log x$. Finally, it follows from (4.2) and (4.3) that $\lim_{r\to\infty} \gamma_r(a)=\lim_{a\to\infty} \gamma_r(a)$ exists. Comparing (4.1) with (1.2) we see that this limit is in fact Euler's constant.

6. *History.* Properties of $h(n)$ in place of $\log n$ have been used to obtain some of Tchebychef's results on the distribution of primes, without recourse to limiting processes [1, 2, 3, 4]. The use of $h(n)$ instead of $\log n$ is sometimes even quite natural. Thus, if $d(n)$ is the number of divisors of n, then [5, p. 262)

$$d(1)+d(2)+\ldots+d(n)=[n/1]+[n/2]+\ldots+[n/n]=nh(n)+O(n).$$

Functions which approximate $\log n$ more closely than $h(n)$, but whose values are rational or algebraic, have been used to give an "elementary" proof of Dirichlet's theorem on primes in arithmetic progressions [6], and to eliminate transcendental methods from the Selberg-Erdös proof of the prime number theorem [7]. The functions studied here are simpler than those used in [6] and [7], but are more widely applicable [8] than $h(n)$.

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J. L. AND L. M.

GLEANINGS FAR AND NEAR

1841. England's prospects for the fourth Test, which begins here on Friday, become harder to estimate every day. The equation is not quite that of two sides' varied strengths just set against each other. It is not even these two estimable strengths plus or minus an unknown quantity against each of them to represent the value to each of a wicket the exact behaviour of which will not be clear until the start of the match. It is an equation which must include playing strength as at present computable, the effect of an uncertain wicket on such strength, and, no less important, the effect of both sides' present morale on such strength and any possible effect anticipations about the wicket may have on that morale. It is a tantalisingly impossible equation of the sort which posed in an algebra book has made schoolboys' lives miserable for generations. —*Manchester Guardian*, January 27th, 1955. [Per Mr. J. Varlow.]

1842. It is only the very great and good who have any living faith in the simplest axioms ; and there are few who are so holy as to feel that 19 and 13 make 32 as certainly as 2 and 2 make 4.—Samuel Butler, *Erewhon*. [Per Mr. R. F. Newby.]

1843. Life would be intolerable if men were to be guided in all they did by reason and reason only. Reason betrays men into the drawing of hard-and-fast lines, and to the defining by language—language being like the sun, which rears and then scorches. Extremes are alone logical, but they are always absurd ; the mean is illogical, but an illogical mean is better than the sheer absurdity of an extreme. There are no follies and no unreasonablenesses so great as those which can apparently be irrefragably defended by reason itself, and there is hardly an error into which men may not easily be led if they base their conduct upon reason only.—Samuel Butler, *Erewhon*. [Per Mr. R. F. Newby.]

THE HARMONIC LOGARITHM

By M. F. Egan.

Recently a good deal of interest has been taken in the properties of numbers which can be expressed and proved without going beyond the number 1 and the Four Rules, in other words by using only rational numbers. Professor T. S. Broderick has established various theorems about the number and distribution of primes, using this "sub-language": (*Journal of the London Mathematical Society*, Vol. 14, pp. 303–310; *Proceedings of the Royal Irish Academy*, 1940, pp. 17–24).

The logarithm is replaced by the "harmonic logarithm", defined thus: If X is the integral part of the positive number $x = X + \alpha$, then

$$Lx = LX = \sum_{r=1}^{X} 1/r, \quad L0 = 0.$$

The following simple results may not be without interest.

I. $$\sum_{1}^{n} Lr = (n+1)Ln - n.$$

Since $1/r$ occurs in Lr, $L(r+1)$, $L(r+2)$, ..., Ln, its coefficient is $n+1-r$. Hence the given sum is equal to $\sum_{1}^{n} (n+1-r)/r = (n+1)Ln - n$.

II. $$\sum_{1}^{n} rLr = \tfrac{1}{2}n(n+1)Ln - n(n-1)/4.$$

We could prove this by the same method, but it is perhaps simpler to note that, since $L(r-1) = Lr - 1/r$, we have

$$r(r+1)Lr - (r-1)rL(r-1) = 2r\,Lr + r - 1,$$

and summing from 1 to n we get the result. By the same method we could sum $r(r+1)Lr$, $r(r+1)(r+2)Lr$, etc.

III. $$2 - 1/Ln > \sum_{1}^{n} 1/rL^2r > 5/3 - 1/Ln. \qquad (L^2r \text{ means } (Lr)^2).$$

We have, for $r > 1$,

$$1/L(r-1) - 1/Lr = 1/rL(r-1)Lr > 1/rL^2r > 1/Lr - 1/L(r+1).$$

Summing from $r = 2$ to $r = n$, adding 1 to each of the three members and noting that $C - 1/L(n+1) > C - 1/Ln$, we get the result.

IV. $$L^2n + 2 - 1/n > 2\sum_{1}^{n} Lr/r > L^2n + 1\tfrac{1}{2} - 1/n.$$

Lemma. $$2 - 1/n > \sum_{1}^{n} 1/r^2 > 1\tfrac{1}{2} - 1/n.$$

This follows from the fact that, for $r > 1$,

$$1/(r-1) - 1/r > 1/r^2 > 1/r - 1/(r+1),$$

on summing as in the last case.

Since $L(r-1) = Lr - 1/r$, we have

$$L^2r - L^2(r-1) = 2Lr/r - 1/r^2,$$

and summing from 1 to n we get the result.

V. $$L_2 n + 2 > \sum_1^n 1/rLr > L_2 n - 2\tfrac{1}{2}.$$ ($L_2 n$ means $L(Ln)$).

This has some importance. Broderick has proved, (in the " sub-language "), that if p stands for a prime number,

$$\sum_{p \leqslant n} 1/p = \sum_1^n 1/rLr + O(1).$$

Accordingly we can write $L_2 n + O(1)$ in the r.h.s.

We define the " interpolation-function " Mx thus: If X is the integral part of $x = X + \alpha$, $Mx = Lx + \alpha/(X+1)$. ($MX = LX = Lx$.)

If $x = Lr = Mr$ (r an integer), $L_2 r = Lx$, $M_2 r = Mx$. We note that

$$L_2 r + \tfrac{1}{2} > M_2 r \geqslant L_2 r. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5.1)$$

Again, $$M_2(r-1) = M(x - 1/r) = M(X + \alpha - 1/r).$$

Two cases arise:

Case 1. $\alpha \geqslant 1/r$.

$$M_2(r-1) = Mx - 1/r(X+1),$$

so that $$1/rx > 1/r(X+1) = M_2 r - M_2(r-1) > 1/r(x+1) > 1/rx - 1/rx^2.$$

Case 2. $\alpha < 1/r$. Here $x - 1/r$ lies between $X - 1$ and X, so that

$$\begin{aligned} M_2(r-1) &= M(X-1) + (1 + \alpha - 1/r)/X \\ &= MX + \alpha/X - 1/rX \\ &= Mx + \alpha/X(X+1) - 1/rX, \text{ whence} \end{aligned}$$

$$\begin{aligned} 1/rX \geqslant M_2 r - M_2(r-1) &= 1/rX - \alpha/X(X+1) > 1/rX - 1/rX(X+1) \\ &= 1/r(X+1) > 1/rx - 1/rx^2. \end{aligned}$$

Also, $1/rX = 1/rx + \alpha/rxX < 1/rx + 1/r^2 xX \leqslant 1/rx + 1/r^2$.

Accordingly we have, in both cases,

$$1/rLr + 1/r^2 > M_2 r - M_2(r-1) > 1/rLr - 1/rL^2 r.$$

Summing from 1 to n, and taking account of IV (lemma), and III, we find

$$\sum_1^n 1/rLr + 2 > M_2 n > \sum_1^n 1/rLr - 2,$$

whence, putting $r = n$ in (5.1), we get the result.

VI. If the integers a, $b \geqslant 3$, 2, then

$$L(a-1) + L(b-1) > L(ab) > La + Lb - 1. \quad \ldots\ldots\ldots\ldots\ldots (6.1)$$

This holds for 3, 2. Also, if it holds for a, b, it holds for a, $b+1$ (and hence for all greater values of a and b). For

$$L(a(b+1)) - L(ab) = \sum_{r=1}^{a} 1/(ab + r).$$

Since each of the a terms on the right lies between $1/ab$ and $1/a(b+1)$, the sum lies between $1/b$ and $1/(b+1)$; hence

$$L(a-1) + Lb = L(a-1) + L(b-1) + 1/b > L(ab) + 1/b > L(a(b+1));$$

and

$$L(a(b+1)) > L(ab) + 1/(b+1) > La + Lb - 1 + 1/(b+1) = La + L(b+1) - 1, \text{ etc.}$$

Again, if $a, b \geqslant 2, 3$, and c is any positive integer,

$$L(a-1)+L(b-1)+L(c-1) > L(abc) > La+Lb+Lc-2,$$

This is evident if $c=1$. If $c \geqslant 2$, we have

$$L(a-1)+L(b-1)+L(c-1) > L(ab)+L(c-1) > L(abc) > L(ab)+Lc-1 > La+Lb+Lc-2.$$

Hence we infer that if two of the integers a_r are $\geqslant 3$, 2 respectively,

$$\sum_1^n L(a_r-1) > L(a_1a_2\ldots a_n) > \sum_1^n La_r - n + 1. \quad \ldots\ldots\ldots\ldots\ldots\ldots (6.2)$$

(The second inequality, though not the first, holds if 2, 2 occur among the a_r; for $L4 > L2 + L2 - 1$, whence the argument proceeds as before. If $n-1$ of the $a_r = 1$, $>$ is replaced by $=$).

Corollary: Using I, we find, for $n > 2$,

$$nLn - n = \sum_1^n L(r-1) > L(n!) > \sum_1^n Lr - n > nLn - 2n. \quad \ldots\ldots\ldots (6.3)$$

Again, *if x_r denotes a rational number $\geqslant 1$, we have*

$$\sum_1^n Lx_r > L(x_1x_2\ldots x_n) \geqslant \sum_1^n Lx_r - n + 1. \quad \ldots\ldots\ldots\ldots\ldots (6.4)$$

Writing X_r for the integral part of x_r, the second part of the statement follows at once from the remark after (6.2); for $\prod_1^n x_r \geqslant \prod_1^n X_r$, and $Lx_r = LX_r$.

To prove the first part, we consider two cases:

(i) If one of the x_r, say x_1, $\geqslant 2$, we write $Y_r = X_r + 1$. Then Y_1, $Y_2 \geqslant 3, 2$, and $Lx_r = L(Y_r - 1)$. Hence, by (6.2)

$$\sum_1^n Lx_r > L(Y_1Y_2\ldots Y_n) > L(x_1x_2\ldots x_n).$$

(ii) If every $x_r < 2$, $Lx_r = 1$. Also, $\prod_1^n x_r < 2^n$, so that its integral part does not exceed $2^n - 1$; and we have to prove that $n > L(2^n - 1)$. This is true for $n = 2$, since $2 > L3$. For $n > 2$, we have $2^n = 4 \times 2^{n-2}$, whence by (6.2) we get

$$L(2^n) < L3 + (n-2)L1 = n - 1/6 < n,$$

which completes the proof.

Most of the results I have given are modifications or developments of results given by Professor Broderick (*l.c.*).

M. F. EGAN.

1844. The quinella totalisator, which is popular overseas, operates on one race and the investor nominates two horses. He is issued with a ticket bearing the two numbers of the horses nominated and to be successful his nominated horses must run first and second or second and first.

The expert's report states that on a quinella totalisator with a field of 10 starters there are 55 combinations and with a field of 24 starters there would be 300 such combinations.—*The New Zealand Herald*, August 17th, 1954. [Per Mr. C. M. Segedin.]

THE POWER OF A POINT FOR A CURVE

By E. H. Neville

To attach, in relation to a curve $f(x, y) = 0$, a geometrical significance to the value of $f(x, y)$ at an arbitrary point of the plane, it is necessary to impose a condition which has the effect of standardising the function $f(x, y)$ by means of some divisor independent of the variables. This article, another outcome of Mr. Durell's query on the semi-cubical parabola,* is concerned with normalisation of the equation of an algebraic curve, and with Laguerre's interpretation of a standard form of $f(x, y)$. The axes are rectangular throughout.

A linear equation $ax + by + c = 0$ is in a normal form if the function $ax + by + c$ measures a normal distance between the line $ax + by + c = 0$ and the point (x, y); the condition to be satisfied is $a^2 + b^2 = 1$. If a and b are real, the condition is equivalent to $|\, a + ib \,| = 1$, but the notion of distance has its place in coordinate geometry with complex elements, and if a and b are complex the conditions $a^2 + b^2 = 1$, $|\, a + ib \,| = 1$ are not related. It is the condition $a^2 + b^2 = 1$ that characterises a normal form of the linear equation, and it will be convenient to call $a^2 + b^2$ the *normal measure* of the function $ax + by + c$. Unless $a^2 + b^2 = 0$, that is, unless the geometry is complex and the line is isotropic, either square root of the normal measure may be used as a normalising divisor of the equation.

The normal measure involves only the coefficients of the terms of the first degree, and is not affected by a change of origin or by a rotation of the axes.

A product of linear factors is measured by the product of the measures of the individual factors. If the normal measure of the product is unity, we can normalise every factor without altering the product, and the product itself is said to be in a normal form. Provided only that the normal measure of the product is not zero, a square root of the normal measure is a normalising divisor of the product.

The normal measure of $\Pi(a_r x + b_r y + c_r)$ is the normal measure of the homogeneous polynomial $\Pi(a_r x + b_r y)$ formed of the terms of highest degree in the original product; this is the basis of our next step. The normal measure of any polynomial $f(x, y)$ is defined as the normal measure of the group of terms of highest degree in $f(x, y)$. Let this group of terms be $\phi(x, y)$, and denote the normal measure of $f(x, y)$ by $M(f)$. The homogeneous polynomial $\phi(x, y)$ can be factorised as $\Pi(a_r x + b_r y)$, and we have

$$\begin{aligned} M(f) = M(\phi) &= \Pi(a_r^2 + b_r^2) = \Pi\{(a_r + ib_r)(a_r - ib_r)\} \\ &= \Pi(a_r + ib_r) \,.\, \Pi(a_r - ib_r) = \phi(1, i) \,.\, \phi(1, -i). \end{aligned}$$

If then

$$\phi(x, y) = C_0 x^n + C_1 x^{n-1} y + C_2 x^{n-2} y^2 + \dots,$$

we have

$$M(f) = A^2 + B^2$$

where $$A = C_0 - C_2 + C_4 - \dots, \quad B = C_1 - C_3 + C_5 - \dots.$$

A formula for the normal measure $M(\phi)$ in terms of the coefficients of $\phi(x, y)$ having been found, there is a good case for making a fresh start with this formula for definition. The application to real geometry is then formally independent of complex variables, and in complex geometry the definition of normal measure is not dependent on the fundamental theorem of algebra. But the relation of the measure of a product to the measures of its factors has now to be established, and if a real interpretation of the normalisation of a quadratic factor is wanted, such an interpretation has to be found.†

* *Math. Gazette*, vol. 37, p. 58; Note 2330.

† See my note, "The asymptotic involution of the ellipse," *Math. Gazette*, vol. 39, p. 226; Note 2542.

Two expressions for A and B in terms of the function $\phi(x, y)$ will be useful. If we write

$$\phi(x, y) = \phi_1(x, y^2) + y\phi_2(x, y^2)$$

where $\phi_1(x, t) = C_0x^n + C_2x^{n-2}t + C_4x^{n-4}t^2 + \dots,$

$$\phi_2(x, t) = C_1x^{n-1} + C_3x^{n-3}t + C_5x^{n-5}t^2 + \dots,$$

we have $$A = \phi_1(1, -1), \quad B = \phi_2(1, -1),$$

and also $$\phi_1(x, -x^2) = Ax^n, \quad \phi_2(x, -x^2) = Bx^{n-1}.$$

Since therefore Ax^n, Bx^{n-1} are the remainders when $\phi_1(x, t)$, $\phi_2(x, t)$, regarded as polynomials in t, are divided by $t + x^2$, we have

$$\phi(x, y) = (x^2 + y^2)\,\bar{\phi}(x, y) + x^{n-1}(Ax + By),$$

where $\bar{\phi}(x, y)$ is a polynomial in x and y. In other words, we can identify A and B from $\phi(x, y)$ by dividing $\phi(x, y)$ by $x^2 + y^2$.

Given two functions $\phi(x, y)$, $\psi(x, y)$ whose product is $\chi(x, y)$, we have

$$\{\phi_1(x, y^2) + y\phi_2(x, y^2)\}\{\psi_1(x, y^2) + y\psi_2(x, y^2)\} = \chi_1(x, y^2) + y\chi_2(x, y^2)$$

where $\chi_1(x, t) = \phi_1(x, t)\psi_1(x, t) + t\phi_2(x, t)\psi_2(x, t),$

$$\chi_2(x, t) = \phi_1(x, t)\,\psi_2(x, t) + \phi_2(x, t)\psi_1(x, t).$$

Hence $$A_\chi = A_\phi A_\psi - B_\phi B_\psi, \quad B_\chi = A_\phi B_\psi + B_\phi A_\psi,$$

and therefore $$M(\chi) = M(\phi)\,M(\psi).$$

From the identification of $x^{n-1}(Ax + By)$ as a remainder it follows that two polynomials of the same degree which differ by a multiple of $x^2 + y^2$ have the same measure. If the axes are rotated, $\phi(x, y)$ becomes a polynomial $\Phi(X, Y)$ in the new variables, and because $x^2 + y^2$ becomes $X^2 + Y^2$, the normal measure of $\Phi(X, Y)$ is the normal measure of the polynomial in X and Y into which $x^{n-1}(Ax + By)$ is transformed. Since the normal measures of the linear functions x and $Ax + By$ are unchanged by rotation, this is true also of the normal measure of the product $x^{n-1}(Ax + By)$, and therefore $\phi(x, y)$ and $\Phi(X, Y)$ have the same normal measure.

A change of origin does not affect the terms of the highest degree in $f(x, y)$, and therefore $M(f)$, calculated from $\psi(x, y)$, is entirely independent of the axes of reference, as long as these remain rectangular, and an equation which is normal in one pair of rectangular coordinates is normal in any other pair.

If the equation $f(x, y) = 0$ is in a normal form, the value of $f(x, y)$ at any point of the plane is called the power of that point for the curve. The name is adopted from the geometry of the circle, in spite of the anomaly that the circle, whose normal measure is necessarily zero, is outside the immediate scope of the definition. If $f(x, y) = 0$ is normal, so also is $-f(x, y) = 0$, and the sign of the power changes when the equation is changed. Ambiguity of some kind is inevitable, and there may be other ways of dealing with it.

In real geometry, $M(\phi)$ is zero only if $\phi(x, y)$ has $x^2 + y^2$ for a factor, and we can embrace bicircular curves in our definition by the extension that $x^2 + y^2$ is to be *the* normal form of this factor. Then $(x^2 + y^2)^k\psi(x, y)$ is in a normal form if and only if $\psi(x, y)$ is in a normal form.

In complex geometry a curve can pass through one of the circular points and not through the other. The general form of $\phi(x, y)$ is

$$(x + iy)^h(x - iy)^k\psi(x, y),$$

but we lose the invariance of the normal form if we require the coefficients in the isotropic factors to have specified values, since rotation through the angle β converts $x+iy$ into $e^{i\beta}x+ie^{i\beta}y$.

One interpretation of the power of a point O for a curve which does not pass through either of the circular points is immediate. If we substitute $p+r\cos\theta$, $q+r\sin\theta$ for x, y in the equation $f(x,y)=0$, we have an equation in r in which the constant term is $f(p,q)$ and the coefficient of r^n is

$$\phi(\cos\theta, \sin\theta).$$

Hence if O is the point (p,q) and $\varpi_0(\theta)$ is the power of O for the curve in the direction θ, that is, the product of the n distances from O to the curve in this direction,

$$f(p,q)=(-)^n\varpi_0(\theta)\phi(\cos\theta,\sin\theta).$$

But if $\phi(x,y)$ is normalised, we can suppose the factor $a_r x+b_r y$ of $\phi(x,y)$ to have the form $x\sin\alpha_r - y\cos\alpha_r$, where α_r is an asymptotic direction, and therefore

$$\phi(\cos\theta,\sin\theta)=\pm\Pi\sin(\alpha_r-\theta).$$

The power of the point O for a non-circular curve in the direction θ is numerically the quotient of two functions; the numerator is independent of θ and is the power of O for the curve; the denominator is independent of O and is the product of the sines of the angles between θ and the asymptotic directions.

This theorem loses vividness, for a real curve, if the asymptotic directions are not real, and Laguerre enunciates a theorem of which it is a limiting case. Consider the $2n$ intersections of the curve $f(x,y)=0$ and the circle $x(x-d)+y^2=0$. If $R=x^2+y^2$, then, on the circle, $x=R/d$, $y^2=R(d^2-R)/d^2$. Hence, if

$$f(x,y)=f_1(x,y^2)+yf_2(x,y^2),$$

the values of R at the points of intersection satisfy the equation

$$\left\{f_1\left(\frac{R}{d},\frac{R(d^2-R)}{d^2}\right)\right\}^2-\frac{R(d^2-R)}{d^2}\left\{f_2\left(\frac{R}{d},\frac{R(d^2-R)}{d^2}\right)\right\}^2=0,$$

and since the degree of this equation in R is $2n$, there are no extraneous roots. The term independent of R is $\{f_1(0,0)\}^2$, and this is $\{f(0,0)\}^2$. The terms in R^{2n} come only from the terms of highest degree in $f_1(x,y^2)$ and $f_2(x,y^2)$, and compose the aggregate

$$\left\{\phi_1\left(\frac{R}{d},-\frac{R^2}{d^2}\right)\right\}^2+\frac{R^2}{d^2}\left\{\phi_2\left(\frac{R}{d},-\frac{R^2}{d^2}\right)\right\}^2,$$

which has the value

$$\left\{A\left(\frac{R}{d}\right)^n\right\}^2+\frac{R^2}{d^2}\left\{B\left(\frac{R}{d}\right)^{n-1}\right\}^2,$$

that is, $(R/d)^{2n}M(f)$. If ϖ is the product of the $2n$ values of R, we have therefore

$$\varpi/d^{2n}=\{f(0,0)\}^2/M(f),$$

and since $M(f)$ is unaltered by rotation of axes, the value of ϖ/d^{2n} is independent not only of the magnitude d of the diameter of the circle, but also of the line on which the centre is situated; in other words, ϖ/d^{2n} has the same value for all circles through the origin.

To replace R, the square of a distance, by an actual distance, involves no more ambiguity than is present already if the function $f(x, y)$ is normalised by means of a square root of $M(f)$, and we have Laguerre's theorem :

If an algebraic curve of degree n does not pass through either of the circular points, the power of a point O for the curve is the product of the distances from O to the 2n points in which the curve is cut by an arbitrary circle through O, divided by the nth power of the diameter of the circle.

Since asymptotes are not introduced, this theorem extends substantially the range over which the power of a point for a curve is defined geometrically in real terms when the geometry is real, but the extension is still incomplete. In the case of the ellipse $x^2/a^2 + y^2/b^2 = 1$, for example, no circle through O cuts the curve in four real points if O is outside the pair of circles

$$b^2\{x^2 + y^2 - (2a^2 - b^2)\}^2 = 4(a^2 - b^2)y^2.$$

There is no change in the algebra as the boundary is crossed, but another interpretation in real terms has to be found if one is wanted.

It is the power which we have been considering that enters into Laguerre's theorems on products of tangents and products of normals.* Laguerre begins by asserting that the ratio which in our notation is $\sqrt{\varpi}/(\frac{1}{2}d)^n$ is the same for all circles through O, and defines the power from this ratio. Holst normalises the equation of the curve before proving the theorems on tangent-products and normal-products. Coolidge's proof of Laguerre's theorem on ϖ/d^{2n} depends much more obliquely on the formula for $M(\phi)$ than the proof given above. E. H. N.

* *Math. Gazette*, Vol, 38, p. 192, where references are given.

1845. I went this far with him : " Sir, allow me to ask you one question. If the Church should say to you, ' Two and three make ten,' what would you do? " " Sir," said he, " I should believe it, and I should count like this : one, two, three, four, *ten*." I was now fully satisfied.—Boswell's Journal for 31st May, 1764 ; *Boswell in Holland* (1952), p. 258. [Per Professor E. H. Neville.]

1846. In requiring all sciences to have the certainty of mathematics, Aristotle imposed on them the limitations of mathematics, which can never gain new knowledge, but only transforms old, and presents it in a new dress.—Sir James Jeans, *The growth of physical science* (1947), p. 94. [Per Mr. J. Dickson.]

1847. He proved to be a reliable counter, but after all mathematics, like St. Vitus Dance, is involuntary.—Robert Kemp, *The Malacca Cane*, p. 17. [Per Miss I. M. Smart.]

1848. The word " power " is equivalent to " physical force ", as when we speak of " horse power ". Exact scientific definitions of force are apt to be unintelligible to the layman, but we know at least roughly what we ordinarily mean when we use the word. We mean the impulse to motion, or the restraint of motion, in material bodies : and we usually measure the amount of force by the amount of change produced in relation to the resistance encountered.—O. C. Quick, *Doctrines of the Creed*, p. 61. [Per Mr. W. Tennant.]

FILM GROUPS

By T. J. Fletcher

Cyclically repeating patterns are a common feature of mathematical films. The study of the symmetries which arise in such configurations has a theoretical interest to mathematicians watching the film, and it also has a practical interest to the film maker, because the symmetries present in a diagram can often be used to reduce the labour of drawing. It does not seem that these symmetries have been commented on before, and in this article we attempt a complete enumeration of the cases which can arise.

Before embarking on the main question we will remark that a group of order eight arises merely in connection with the way a film is put in to a projector ; because as well as the correct way it may be put in upside down, inverted right-left, or run backwards. The results of these transformations on the phenomena which the film displays are often far from obvious. Most mathematical films are invariant under them in the sense that they continue to display the same phenomena, but the Fairthorne and Salt films on differential equations are not quite so simple. A full discussion involves careful consideration of the sign conventions and labelling of the diagram, and reveals a great deal about the film which might otherwise pass unnoticed.

This article is concerned with invariance in another sense. A mathematical film displays a two-dimensional diagram dependent on a time parameter, so we consider transformations under which cyclically repeating diagrams reoccupy the same regions in space and time. If the horizontal and vertical axes on the screen are labelled y and z the structure of the film can be specified by three coordinates (t, y, z) ; and if the t coordinate is thought of as a spatial coordinate x we obtain a static three-dimensional representation whose symmetries in space are the same as the symmetries of the film in space and time. If this is done any diagram can be assigned to a particular symmetry group, and the known information about space groups can be reinterpreted to enable a complete enumeration of the relevant space-time groups to be carried out. This paper deals completely with the discrete groups which occur. Continuous groups or semi-continuous groups are not considered.

A full account of the theory of space groups can be found in the standard references (1, 2, 3), but it may be summarised as follows.

I. *Strip Patterns in a Plane.* The pattern consists of two-dimensional motives in the z-plane, arranged on a one-dimensional lattice extending along the x-axis. The symmetry operations involved in addition to translations along the lattice are :

(i) rotations through 180° about axes perpendicular to the z-plane ;
(ii) reflections or glide-reflections in the x-axis ;
(iii) reflections in lines in the z-plane parallel to the y-axis.

Seven groups arise from different combinations of these operations. That is there are only seven basically different designs of this type. They are shown in their simplest forms in figure 1.

II. *Strip Patterns in Relief.* The symmetry operations involved in addition to translations are :

(i) rotations or screw-rotations through 180° about the x-axis ;
(ii) rotations through 180° about axes in the z-plane parallel to the y-axis ;
(iii) rotations through 180° about axes in the y-plane parallel to the z-axis ;
(iv) reflections in points on the x-axis (inversions) ;
(v) reflections in planes parallel to the x-plane ;
(vi) reflections or glide-reflections in the y-plane ;
(vii) reflections or glide-reflections in the z-plane.

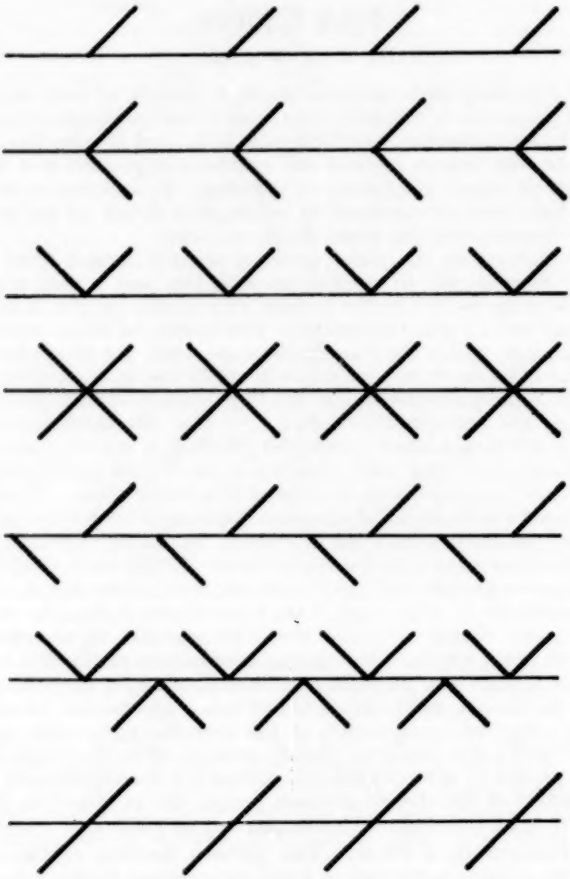

FIG. 1.

31 groups arise from different combinations of these operations. That is, there are just 31 basically distinct strip ornaments in relief. These are illustrated in (1).

III. *Plane Patterns.* The pattern is based on a two-dimensional lattice in the z-plane. There are five types of lattice which give rise to four systems of symmetry (monoclinic, orthorhombic, tetragonal and hexagonal—including trigonal). The symmetry operations involved in addition to the translations of the lattice are :

(i) rotations about two-, three-, four- or sixfold axes parallel to the z-axis ;
(ii) reflections or glide-reflections in lines in the z-plane .

These give rise to just 17 groups which were first described by Polya and Niggli in 1924.

IV. *Plane Patterns in Relief.* The symmetry operations involved in addition to the translations of the lattice are :

(i) rotations about two-, three-, four- or sixfold axes parallel to the z-axis ;
(ii) two-fold rotations or screw-rotations about lines in the z-plane ;
(iii) reflections or glide-reflections in planes perpendicular to the z-plane ;
(iv) reflections or glide-reflections in the z-plane ;
(v) reflections in points in the z-plane (inversions).

These give rise to just 80 groups which were first described by Weber and by Alexander and Hermann in 1929.

V. *Space Patterns.* There are 14 types of three-dimensional lattice arranged in seven systems (six according to some writers). The symmetry operations, which we do not list here, give rise to 230 groups arranged in 32 classes. These groups, which are the basis of theoretical crystallography, were first enumerated independently by Fedorov in 1890, Schoenflies in 1891 and Barlow. They are listed in (3).

VI. *Prismatic Patterns.* To complete an enumeration of the space groups which have applications to films it is necessary to add the groups associated with repeating patterns on a regular n-sided prism. The axis of the prism will be taken as the x-axis.

The symmetry operations involved in addition to translations are :

(i) rotations or screw-rotations about the x-axis ;
(ii) rotations through 180° about axes perpendicular to the x-axis ;
(iii) reflections in planes through the x-axis, or reflections in them with a glide in the direction of the x-axis ;
(iv) reflections in planes perpendicular to the x-axis ;
(v) reflections in points in the x-axis (inversions).

These groups do not seem to have been described in the literature. They can be enumerated in the case of $n = 3$, 4, and 6 by using the tables of the trigonal, tetragonal and hexagonal classes of the 230 space groups and finding the sub-groups which leave the axis invariant. That is to say it is necessary to eliminate those groups with secondary screw axes or with glides with a component perpendicular to the principal axis, and also those which differ from some other group in the class merely by being on a different lattice, or by being differently oriented on the same lattice. This reduces the numbers of groups in these classes from 25, 68 and 27 to 11, 19 and 23 respectively. When the lists of these are studied the law of formation becomes apparent, and it can be seen that the number of groups of this kind is $2n + 5$ when n is odd and $2n + 11$ when n is even. In the odd case there are $n - 1$ pairs which differ only in the sense of the screw-rotation, and in the even case there are $n - 2$ pairs. This distinction matters in crystallography, but in the present application it is not worth preserving. So on this understanding the number of different groups associated with the n-sided prism is $n + 6$ when n is odd, and $n + 13$ when n is even.

It now remains to interpret all the above space groups as symmetry groups in space and time. When this is done some have to be eliminated as they involve symmetries which would interchange time and space axes, and this operation is obviously not permissible in the present context.

I. If the strip patterns of section I are drawn through the screen in such a manner that the x-axis is normal to it they give rise to cyclically repeating linear patterns with differing symmetries about the centre of the screen. The symmetry operations enumerated above have to be re-interpreted as follows :

(i) reversal of the time scale coupled with reflection in the centre of the screen ;
(ii) reflection in the centre of the screen, or reflection together with a shift in time ;

(iii) reversal of the time scale.

These patterns are the simplest that can occur in a moving diagram, and there are only seven of them.

II. The symmetry operations of this section can be re-interpreted to give the groups of cyclic two-dimensional diagrams which have symmetries about horizontal and vertical axes in the screen. These 31 are of great practical

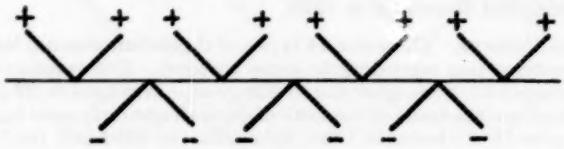

FIG. 2.

importance because diagrams of this type occur very frequently. The importance is greatest when the symmetries involve glides in time, because then the same drawing can be used again (turned if necessary) at different stages in the cycle. Examples of this type of symmetry can be seen in the film *Plucked Strings* where the diagrams have the symmetries of the group numbered 26 in the list given in reference [1], where the group is illustrated by figure 2.

The Fairthorne and Salt films mentioned above also contain symmetries of this type.

III. When one of the axes of the plane patterns described in sections III and IV is distinguished as a time axis we must omit those groups in which there are symmetries which interchange it with other axes. This excludes the ten groups based on rhombic, tetragonal and hexagonal lattices, and so only the seven monoclinic and orthorhombic groups remain. These give rise to seven linear patterns which are not only cyclic in time but which also repeat along the space axis. They are, apart from the addition of the translational symmetries, essentially the same as the groups in section I. The difference between the " monoclinic " and " orthorhombic " diagrams is that the former drift across the screen whereas the latter do not.

IV. For the same reasons as before 41 of these 80 groups must be omitted and so 39 remain. These groups are associated with patterns which are cyclic in time and which possess an axis across the screen along which they repeat, and in which they may be reflected or glide-reflected (the glide taking place in space or time or both).

Some aspects of these symmetries are easily missed, and an animator can multiply his work by a large factor if he fails to recognise them. A film diagram does not move continuously ; it jumps 16 or 24 times a second. Consequently unsuspected subtleties arise from stroboscopic phenomena. This means in practice that the unit cell of the diagram (considered in space and time) is often very much smaller than is at first apparent, and if the animator is not careful he will be animating on a multiply primitive lattice instead of a simply primitive one. An example of this is provided by the diagrams appearing in the Dance-Kaufmann loop film entitled *Rotational, Longitudinal and Transverse Waves*. This shows a line of dots oscillating so that a wave travels along them according to the equation

$$z = \cos(ny - mx).$$

(It will be remembered that " x " and " t " are synonymous in our notation.) The scale may be chosen so that y takes integral values for the different dots in the diagram. If y increases by unity and x (the time) increases by n/m the

configuration is repeated in a different position. This is most easily brought about by moving the drawing along its axis. If the quantity n/m is made equal to the time interval between frames of the film the eye sees something very different from a drawing moving with uniform velocity from left to right. The eye strobes the picture and sees a line of dots moving up and down sinusoidally. *There is relative movement between the parts of the diagram in spite of the fact that the photographs are of a mechanically rigid drawing.* The Dance-Kaufmann film is made on this principle but employs four or five drawings used in rotation, and it shows a cyclic phenomenon of much longer period than just the four or five frames. To see this effect it is not necessary to make a film. The single drawing can be made into a Zoetrope (4), and with a suitable spacing of the dots the required result is produced.

V. When the three-dimensional space groups are interpreted as groups in space and time it is necessary to omit the 36 groups in the five classes of the cubic system as they involve interchanges of space and time axes. This leaves 194 groups associated with cyclic configurations which also repeat in a lattice over a plane. No specifically mathematical films display this type of symmetry so far, but it can sometimes be seen in the animated backgrounds to advertisements in the commercial cinema.

VI. The groups in this section correspond to film diagrams of polygonal patterns with rotational symmetries about the centre of the screen. The French film *Polygones Reguliers* consists almost entirely of diagrams of this type. We have shown that there are just $n+6$ of them when n is odd ($\geqslant 3$), and $n+13$ when n is even ($\geqslant 4$).

One cannot watch mathematical films for long without noticing that they somehow present a geometry with a feeling which is different from that of the static geometry with which we are more familiar. The main reason for this is to be found in the symmetry groups on which they are based. Klein gave a celebrated answer to the question " What is Geometry? " saying that it is the study of invariants under the transformations of a group. Mathematical film making is both a science and an art. The symmetry groups are the key to both the structure of the science and the aesthetics of the art.

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1849. SIR—Mr. Claydon (March 26th issue) objects to the lucid graphs with which you illustrate many articles, thus benefiting the less literate readers like myself, who are unable to analyse visually the figures given in the " less frivolous third " of *The Economist.* Many people will agree that the accuracy lost in graphical presentation is compensated for by clarity and impact. If one wishes to quote statistics with authority, it is perfectly possible—and preferable—to examine them in full in an official source such as the " Monthly Digest ".

After all, are we entirely to forgo pictorial methods? Now deprived of our Horror Comics, must we also lose our graphs?—Yours faithfully, A. R. Wolfe. —Letter in *The Economist.* [Per Mr. E. H. Lockwood.]

A DIGIT TRANSFORMATION

By Eric Goodstein

We describe a variant of the game with digits discussed in a previous article.* There we considered a transformation of any number in a scale $s+1$ in which each digit is replaced by a digit equal, *modulo* s, to the sum of the remaining digits. Starting for instance with the number 16427, in the scale of 10, the first digit remains 1, since $1 = 6+4+2+7 \pmod 9$, the second digit is replaced by 5, since $5 = 1+4+2+7 \pmod 9$, and so on, forming the number 15794; repeating the transformation we obtain in turn 73184, 72461, 49751, 48137, 16427, the original value recurring after six transformations. We showed that under this transformation any number x, with m digits in the scale $s+1$ (none equal to s) recurs unless $m-1$ has a common factor (greater than unity) with the quotient of sm by the highest common factor of sm with the sum of the digits of x.

We consider now the transformation in which each digit of a number is added to the next to form the new digit, the final digit being added to the first to form the new final digit (each sum being replaced by a digit which is its equal *modulo* the greatest digit). Thus starting with the number 325, in the scale of 10, we form in turn 578, 364 and so on, and after eighteen transformations the original 325 recurs. We shall show that every number with an odd number of digits recurs; for numbers with an even number $2k$ of digits we prove that, at the latest, the $(2k-1)$th transform recurs (the number 10^s showing that the value $2k-1$ is the best possible). Let

$$a, b, c, \dots, j, k$$

be the N digits of a number in the scale $s+1$. After one transformation the digits are equal (mod s) to

$$a+b, \quad b+c, \dots, j+k, \quad k+a;$$

denote the digits (mod s) obtained after n transformations by

$$p_n a + q_n b + \dots + y_n j + z_n k, \quad p_n b + q_n c + \dots + z_n a, \dots, \quad p_n k + q_n a + \dots + z_n j,$$

so that

$$p_{n+1} = p_n + z_n, \quad q_{n+1} = q_n + p_n, \dots, \quad z_{n+1} = z_n + y_n$$

whence

$$\Delta p_n = z_n, \quad \Delta q_n = p_n, \dots, \quad \Delta z_n = y_n$$

and therefore $p_n, q_n, \dots, z_n$ all satisfy the difference equation

$$\Delta^N \xi_n = \xi_n.$$

The values of $p_n, q_n, \dots$ for $n = 0, 1, 2, \dots, N-1$ are (taken *modulo* s) given by the rows of the array

$$\begin{array}{l} 100\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 0 \\ 110\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 0 \\ 121\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ 1\binom{N-1}{1}\binom{N-1}{2}\ldots\ldots\ldots\binom{N-1}{N-2}1 \end{array}$$

in which the columns are the coefficients of x^r, $r \leqslant N-1$, in the expansions of

* *Gazette*, Vol. XXV, No. 265, 1941, pp. 156–159.

the fractions

$$1/(1-x),\quad x/(1-x)^2,\quad x^2/(1-x)^3,\ \ldots\ldots\ldots\ldots,\quad x^{N-1}/(1-x)^N.$$

Hence, since

$$\frac{x^{r-1}(1-x)^{N-r}}{(1-x)^N-x^N}-\frac{x^{r-1}}{(1-x)^r}=\frac{x^{N+r-1}}{(1-x)^r\{(1-x)^N-x^N\}},$$

the expansions of $x^{r-1}(1-x)^{N-r}/\{(1-x)^N-x^N\}$ and $x^{r-1}/(1-x)^r$ agree up to the term in x^{N+r-2} (at least) and therefore $p_n, q_n, \ldots, z_n$ are equal (mod s) to the coefficients of x^n in the expansions of the fractions

$$\frac{(1-x)^{N-1}}{(1-x)^N-x^N},\quad \frac{x(1-x)^{N-2}}{(1-x)^N-x^N},\quad \frac{x^2(1-x)^{N-3}}{(1-x)^N-x^N},\ \ldots\ ,\quad \frac{x^{N-1}}{(1-x)^N-x^N},$$

respectively.

When we divide $(1-x)^N-x^N$ (arranged by increasing powers of x) into unity there are at most s^N different remainders (mod s) and so after at most s^N+1 steps a remainder recurs. If the first remainder which recurs is the μth and if it recurs as the $(\mu+\nu)$th remainder then

$$\frac{1}{(1-x)^N-x^N}=c_0+c_1x+\ldots+c_{\mu-1}x^{\mu-1}+\frac{x^\mu}{1-x^\nu}(c_\mu+c_{\mu+1}x+\ldots+c_{\mu+\nu-1}x^{\nu-1})$$

so that

$$(1-x^\nu)/\{(1-x)^N-x^N\}=c_0+c_1x+\ldots+c_{\mu+\nu-1}x^{\mu+\nu-1}-(c_0+\ldots+c_{\mu-1}x^{\mu-1})x^\nu$$

which shows that $(1-x^\nu)$ is divisible by $(1-x^N)-x^N$ (mod s), from which it follows that in the expansion of $1/\{(1-x)^N-x^N\}$ there are in fact no non-recurrent terms and the first ν coefficients recur in a block.

Accordingly the expansions of each of the fractions

$$x^{r-1}(1-x)^{N-r}/\{(1-x)^N-x^N\},\quad r=1, 2, \ldots, N,$$

recur (mod s) from the term in x^{N-1} (at the latest) with a period of ν terms, so that each of $p_N, q_N, \ldots$ recurs. (The recurrence may start earlier as is shown by the number 10^3, the *first* transform of which recurs). If however N is odd, each of $p_0, q_0, \ldots$ recurs : for if

$$1/\{(1-x)^N-x^N\}=c_0+c_1x+\ldots+c_{\nu-1}x^{\nu-1}+c_0x^\nu+c_1x^{\nu+1}+\ldots,$$

then, since $(1-x)^N-x^N$ divides $1-x^\nu$ (mod s),

$$c_{\nu-1}=c_{\nu-2}=\ldots=c_{\nu-N+1}=0$$

and so if we multiply the expansion of $1/\{(1-x)^N-x^N\}$ by any polynomial of degree less than N, and if the product is Σd_nx^n then, for any integers $r\geqslant 1$ and $s<\nu$,

$$d_{r\nu+s}=d_s$$

showing that the first block of ν digits recurs ; in particular this is true if we multiply by $x^{r-1}(1-x)^{N-r}$ for any $r=1, 2, \ldots, N-1, N$, which completes the proof. E. G.

1850. It was obvious then that an improvement was on the way, for the mercury had been steadily rising since the previous week-end. In moments of despair I wonder if the professional weather men would not be better off with less mathematics and more common sense.—*Daily Telegraph.* [Per Mr. H. G. Woyda.]

THE PROBLEM OF THE THIRTEEN SPHERES

By John Leech

It was conjectured by David Gregory* that a sphere can touch thirteen non-overlapping spheres equal to it. In a recent paper† proofs have been given that this in fact is impossible. In the present paper I outline an independent proof of this impossibility, certain details which are tedious rather than difficult being omitted.

We consider the configuration of the points of contact of the sphere with those around it, or, equivalently, the configuration (on a sphere) of the centres of these spheres. In terms of this configuration, we prove that there cannot be thirteen points on a sphere such that the (great circle) distance between any two of them is at least $\frac{1}{3}\pi$. (We assume throughout that the sphere is of unit radius and that distances are measured on the surface of the sphere.)

Let us consider the network on the sphere formed by joining every pair of the points of the configuration whose distance apart is less than $\cos^{-1}\frac{1}{7}$. No two joins of this network cross, since any four points of the network form a quadrilateral of sides at least $\frac{1}{3}\pi$ whose diagonals cannot both be less than $\frac{2}{3}\pi$, the extreme case being that of the regular quadrilateral of side $\frac{1}{3}\pi$ whose diagonals are both exactly $\frac{2}{3}\pi$. The network therefore divides the sphere into polygons whose vertices are the points of the configuration. We may assume that the network is connected, since if it is in detached parts, these may be moved so as to bring vertices of each part within $\cos^{-1}\frac{1}{7}$ of another, and that no point of the configuration is within $\cos^{-1}\frac{1}{7}$ of only one other point, since any such point can be moved until it is within $\cos^{-1}\frac{1}{7}$ of another. Every angle of these polygons exceeds $\frac{1}{3}\pi$, the lower bound, unattained in the network, being for the spherical triangle of sides $\frac{1}{3}\pi$, $\frac{1}{3}\pi$, $\cos^{-1}\frac{1}{7}$, whose angles are $\frac{1}{3}\pi$, $\frac{1}{3}\pi$, $\cos^{-1}(-\frac{1}{7})$. Thus at most five joins of the network meet in any one vertex. The proof consists of showing by consideration of areas that for such a network to exist having thirteen vertices, it must divide the sphere into triangles except possibly for one quadrilateral, and that for topological reasons such networks do not exist.

The least area of a triangle of the network is that of an equilateral triangle of side $\frac{1}{3}\pi$. This has angles of $\cos^{-1}\frac{1}{3}$ and its area is $3\cos^{-1}\frac{1}{3}-\pi=0{\cdot}5513$.

The least area of a quadrilateral of the network is that of an equilateral quadrilateral of side $\frac{1}{3}\pi$, one of whose diagonals is of length $\cos^{-1}\frac{1}{7}$, this is of area $2(\cos^{-1}(-\frac{1}{3})+\frac{2}{3}\pi-\pi)=1{\cdot}334$.

The least area of a pentagon of the network is that of an equilateral pentagon of side $\frac{1}{3}\pi$ having two coterminous diagonals of length $\cos^{-1}\frac{1}{7}$, and area 2·226.

Thus the excess of the area of a quadrilateral over the minimum for two triangles is at least 0·231, and of a pentagon over three triangles at least 0·572. It is clear further that for $n\geqslant 5$ the excess of the area of an n-gon over the minimum area of $(n-2)$ triangles increases with n, since if on a side of any n-gon is abutted a triangle of minimum area, the resulting $(n+1)$-gon has to be deformed such that the original common side becomes a diagonal of length not less than $\cos^{-1}\frac{1}{7}$ with consequent increase of area.

By Euler's theorem on polyhedra,

$$V+F=E+2,$$

i.e.
$$\begin{aligned}2V-4&=2E-2F\\&=3F_3+4F_4+5F_5+\ldots-2(F_3+F_4+F_5+\ldots)\\&=F_3+2F_4+3F_5+\ldots,\end{aligned}$$

* In an unpublished notebook at Christ Church, Oxford.

† K. Schütte and B. L. van der Waerden, *Math. Annalen* 125 (1953) 325–334.

where F_n is the number of n-gons of the network. The area of the network exceeds

$$\begin{aligned}&0{\cdot}5513F_3 + 1{\cdot}334F_4 + 2{\cdot}226F_5 + \ldots\\ &= 0{\cdot}5513(F_3 + 2F_4 + 3F_5 + \ldots) + 0{\cdot}231F_4 + 0{\cdot}572(F_5 + \ldots)\\ &= 0{\cdot}5513(2V - 4) + 0{\cdot}231F_4 + 0{\cdot}572(F_5 + \ldots).\end{aligned}$$

Now the area of the sphere is 4π, whence

$$\begin{aligned}2V - 4 &\leqslant 4\pi/0{\cdot}5513\\ &= 22{\cdot}8,\end{aligned}$$

and

$$V \leqslant 13.$$

Suppose now, if possible, that $V = 13$. Then $2V - 4 = 22$, and we have

$$4\pi \geqslant 0{\cdot}5513 \times 22 + 0{\cdot}231F_4 + 0{\cdot}572(F_5 + \ldots),$$

i.e.

$$0{\cdot}438 \geqslant 0{\cdot}231F_4 + 0{\cdot}572(F_5 + \ldots).$$

Thus $F_4 = 0$ or 1 and $F_5 = \ldots = 0$, and the network has to divide the sphere into triangles except possibly for the one quadrilateral.

In the case $F_4 = 0$, we have $F_3 = \frac{2}{3}E$, and $13 + \frac{2}{3}E = E + 2$, giving $E = 33$ and an average number of $66/13 > 5$ joins meeting in each vertex, which is contrary to requirements.

In the case $F_4 = 1$, Euler's theorem gives $F_3 = 20$, $E = 32$, and 4 joins meet at some one vertex and 5 at each other. Despite the consistency of these numbers, there is in fact no polyhedron which has them. I know of no better proof of this than sheer trial. One attempts to construct the plane net of such a polyhedron beginning with the quadrilateral and assuming in turn that the vertex at which only four joins meet is or is not a vertex of the quadrilateral. In either case a triangle is abutted on to each side of the quadrilateral, and a fifth join drawn to 3 or 4 of the vertices of the quadrilateral in the respective cases. It is immediately found to be impossible to complete the network as required if the further ends of these joins and the further vertices of the triangles are not all distinct points, and very shortly afterwards to be impossible even if they are. This completes the proof.

The question of when one can construct nets having prescribed numbers of vertices and numbers of joins meeting in them is of some independent interest. Among other cases of impossibility are : 7 vertices, at most 4 joins meeting at each vertex, all triangles except for one quadrilateral or pentagon ; 13 vertices, at most 5 joins meeting at each vertex, all triangles except for one pentagon. Among cases of possibility are : 13 vertices, at most five joins meeting at any one vertex, all triangles except for one hexagon or two quadrilaterals (the latter in several ways, in one of them the quadrilaterals abut and removal of their common side leaves the former), but the former of these is not the net of a true polyhedron as the hexagon has two sides in common with one of the triangles. There is no obvious criterion for whether it is possible to construct a net having prescribed regions and prescribed numbers of joins meeting in each vertex when these prescriptions are consistent with Euler's theorem. (In all these nets the region outside the net is counted as a region of the network ; for practical reasons this region is usually chosen not to be one of the triangles.) J. L.

AN ANALYSIS OF ERRORS APPEARING IN A TEST ON ALGEBRAIC FRACTIONS

BY THE BOARD OF STUDIES FOR MATHEMATICS*

University of Birmingham Institute of Education

A series of investigations is being planned by a section of the Board of Studies for Mathematics in the Institute of Education of the University of Birmingham, with the object of gaining information about the types of difficulty experienced by children in various branches of grammar school mathematics, and the types of error made. An analysis of these errors, both from the mathematical and psychological points of view, is highly informative and relevant to a discussion of teaching methods in the subject. It is hoped that later it will be possible to prepare standardised tests which would be of diagnostic value to teachers. The co-operation of other teachers would be welcomed in furthering this investigation, either in the form of comments or criticisms in the light of their own experience, or in helping to widen the sample of children tested in future work.

This article reports on the first investigation of the series, in which a test on algebraic fractions (shown in Table 1) was given first to a pilot group of about 250 children and later to about 1000 children in ten grammar schools in Birmingham and district.†

Administration and Procedure

The test was planned to be done in one school period so as not to interfere with other school arrangements. The forms had 35–45 mins. for it, and this proved enough for the majority of the children, some taking as little as 15–20 mins. A small proportion of children, however, did not finish, but the effect of this fact on the analysis of the results will not have been serious, for reasons implicit in the later discussion.

It was found that in different schools the topic of algebraic fractions was taught at widely different stages in the course—as early as the second year in some and not until the fourth (pre-certificate) year in others. In addition, the educational trend which has found expression in the Alternative Syllabus in mathematics has caused less attention to be paid to this topic than formerly, and there thus seemed a risk that the pupils at some schools would have had little deliberate teaching in it. However, only one school among those tested avowedly followed the Alternative Syllabus, and moreover the questions set cannot be regarded as outside the scope of that syllabus, except, perhaps, for a few near the end.

In view of these differences between schools, it was decided not to test anyone below the fourth year or below the age of 14 years, and the test took place in the March of that year. Actually, these limits were not observed strictly. A few children just below 14 years were found among older children in the groups tested, and regarded as being at a comparable stage of development. Again, in one school the top stream misses a form and reaches the Fourth Form in its third year. These boys were also regarded as at a com-

* The members of the Board of Studies responsible for the work were Mrs. N. M. Barnett, Mr. E. C. Cull, Mr. W. O. Storer and Dr. G. E. R. Burroughs. The bulk of the analysis was carried out by Mrs. Barnett and Mr. Storer; Mr. Storer was also responsible for the final collation and the writing of the present report.

† We take this opportunity of thanking the Head Teachers and Assistant Teachers at the schools concerned, for their kindness in permitting us to use some of their forms for this purpose, and for their interest and co-operation in the administration of the test.

parable stage to the others. In one school the certificate year forms took the test as well as the pre-certificate year, but in the others this was not possible. For these reasons the results offered here are not claimed as standardised for a representative sample of a particular age group. The widely divergent results from different schools render meaningless the concept of a norm in this matter, and any figures given later are indicative rather than absolute.

Method of Analysis

The marking of the pilot test revealed a much poorer performance than had been expected, with many children obtaining less than 10 right out of 52. Even with children who scored over 30, many of the errors revealed serious misconceptions. It was therefore concluded that the weaker children had such a scanty acquaintance with the subject that their answers were likely to be largely random, and an attempted analysis of their errors would not be significant. The mark of 20 was chosen as a dividing line, and those papers with scores below 20 were not analysed. Even from those with scores of 20 and over (numbering 539 out of 1053 from nine schools) quite enough material has been obtained for discussion. In the tenth school all except one of the 81 children scored less than 20. It was thought that this poor result might indicate the presence in that school of abnormal conditions of childrens' ability or of the organisation of the teaching, and so these papers were not included in the final figures.

Each wrong answer was examined and any errors made in it were classified. 20 categories of error were used but in the presentation of results here some have been grouped together to give the following 15, which are illustrated by examples and given letters for reference.

A. Denominator wrong.

Q. 27 : $\frac{1}{b}+\frac{1}{d}=\frac{1}{b+d}$.

Q. 16 : $\frac{1}{a+b}+\frac{3}{a+b}=\frac{2}{a+b}$,

presumably by adding numerators and denominators.

Q. 31 : $\frac{a}{b}+\frac{2}{b+c}=\frac{ac+2}{b+c}$.

Q. 34 : $\frac{2a}{a+b}+\frac{4c}{a+4c}=\frac{8ac+4bc}{a+b+4c}$.

B. Numerator wrong.

Q. 27 : $\frac{1}{b}+\frac{1}{d}=\frac{2}{bd}$. Q. 30 : $\frac{a}{b}+\frac{c}{d}=\frac{ab+cd}{bd}$.

C. " Partial " Cancelling (and a few other cancelling errors).

Q. 6 : $\frac{3}{6a+b}=\frac{1}{2a+b}$. Q. 7 : $\frac{3a-b}{b}=3a-1$.

Q. 31 : $\frac{a}{b}+\frac{2}{b+c}=\frac{a(b+c)+2b}{b(b+c)}=a+2$.

Other forms of wrong cancelling were originally given separate categories,

e.g. Q. 8 : $\frac{3a+3b}{a+b}=3+3=6$;

striking out a common term without leaving a trace as in Q. 7 above and in Q. 4 : $\frac{a^2}{ab}=\frac{2}{b}$; also in particular $\frac{a}{a}=0$ in Q. 36 and 41.

It became evident, however, that in the hands of an expert at the game partial cancelling runs riot and breaks the bounds of logical categories. A particularly bad example was

$$\text{Q. 34 :}\quad \frac{2a}{a+b}+\frac{4c}{a+4c}=\frac{2a^2+8ac+4ac+4bc}{a^2+ab+4ac+4bc}=\frac{2+8ac}{ab}.$$

D. Faulty Cancelling involving Subtraction.

$$\text{Q. 46 :}\quad \frac{5a}{a}=4a. \qquad \text{Q. 11 :}\quad \frac{3b-3a}{a-b}=\frac{3}{-1}=2.$$

Although only a few examples of this occurred, it seemed significant enough to be retained as a separate category.

E. Errors in Inversion.

$$\text{Q. 39 :}\quad \frac{a}{b}\times\frac{c}{d}=\frac{ad}{bc}. \qquad \text{Q. 42 :}\quad a\div\frac{a}{b}=\frac{1}{b}.$$

A few errors apparently in this category may be ascribable to other causes, e.g. with Q. 5 :

$$\frac{6}{6a+6b}=a+b$$

is probably a case of faulty cancelling, striking out the 6 without trace.

F. Inability to deal with the type $a+\frac{b}{c}$.

Although sympathy was felt with those children who regarded the expressions in Q. 12, 13, 14 as already in their most convenient forms, the instruction had been to bring them to a single fraction.

G. Overlooking a common monomial factor.

H. Overlooking a common binomial factor.

I. Errors in either inserting or removing brackets, including the omission of brackets and wrong factorising.

Q. 47 : $a(a+b)=a^2+b.$ Q. 18 : $-3(a+b)=-3a+3b.$

Q. 37 : $(a+b)(a+c)=a^2+bc.$ Q. 47 : $3a+b=3(a+b).$

J. Errors with $(a-b)$ and $(b-a)$, including overlooking their relationship.

K. Omitting the denominator (presumably through confusion with equations).

$$\text{Q. 28 :}\quad \frac{1}{b}-\frac{3}{d}=d-3b.$$

L. Confusion of multiplication with addition.

$$\text{Q. 12 :}\quad a+\frac{1}{b}=\frac{a}{b}. \qquad \text{Q. 39 :}\quad \frac{a}{b}\times\frac{c}{d}=\frac{ad+bc}{bd}.$$

Combined with other errors it appears in

$$\text{Q. 52 :}\quad \frac{abc}{a+b+c}\times\frac{a+b}{a+c}=\frac{abc(ab+b^2)}{a+b+c},$$

in which presumably a " common denominator " $a+b+c$ has been chosen and the second numerator has been multiplied by the missing term b in its denominator.

M. Failure to add like terms. (Tidying up—a venial fault).

$$\text{Q. 16}: \quad \frac{1}{a+b}+\frac{3}{a+b}=\frac{1+3}{a+b}.$$

N. Miscellaneous minor types of error, or unaccountable errors.

$$\text{Q. 45}: \quad a^2\times a^3=a^6 \qquad \text{Q. 39}: \quad \frac{a}{b}\times\frac{c}{d}=\frac{abcd}{bd}.$$

What seemed a complete misconception of multiplication of fractions as exemplified in this Q. 39 was included here, although it may well arise from a confusion of $\times$ and $+$, through

$$\frac{ad+bc}{bd}.$$

O. Pure slips of copying a letter or sign wrongly.

One answer might contain as many as four types of error. For example, three types appear in

$$\text{Q. 48}: \quad \frac{a^2+b}{a+3b}\times\frac{2a+6b}{a^2+3}=\frac{a^4+3b}{3a+18b^2},$$

viz. (1) wrong inversion (or perhaps cross-multiplication) ;

(2) wrong multiplication of brackets $(a^2+b)(a^2+3)=a^4+3b$,

(3) an addition-multiplication confusion in $(a+3b)(2a+6b)=3a+18b^2$.

Errors in categories E, I and L would therefore be noted, but not in H (overlooking a common binomial factor) because the inversion error precluded this.

It was sometimes difficult to decide on the category of an error. In Q. 47 a frequent error was

$$\frac{3a+b}{a+b}=3,$$

sometimes clearly arising by partial cancellation, sometimes clearly by faulty bracketting $3(a+b)$, but often with uncertainty which of these misapprehensions had prevailed. Likewise in Q. 49,

$$\frac{a^3+b^2}{a^2+b}=a+b$$

may arise either from partial cancelling or from wrongly factorising

$$a^3+b^2=(a^2+b)(a+b).$$

The child's working had not been asked for though it was sometimes shown or could be divined. Even if it had been asked for and unlimited time allowed, there is no certainty that every revealing step would have been recorded. With practice, however, most of the errors could be accounted for convincingly,

$$\text{e.g. Q. 33}: \quad \frac{a}{a+b}+\frac{e}{c+d}=a+e$$

was supplied with the intermediate step

$$\frac{a(c+d)+e(a+b)}{(a+b)(c+d)},$$

to which partial cancelling is readily applicable. Some inconsistency in recording these doubtful errors is likely, however, from the fact that the analysis was shared by the two collaborators, whose judgment in obscure cases may not have agreed.

Examples of Significant or Unusual Errors Observed (omitting any already given in illustrating the Categories).

Q. 5 : $\dfrac{6}{6a+6b}=6\left\{\dfrac{1}{a+b}\right\}$.

The answer $\dfrac{1}{2a+2b}$ presumably came via $\dfrac{30+3}{6a+6b}$ and partial cancelling.

Q. 8 : $\dfrac{3a+3b}{a+b}=\dfrac{6ab}{ab}$, presumably from $\dfrac{3a}{a}+\dfrac{3b}{b}=\dfrac{3ab+3ab}{ab}$.

Q. 10 : $\dfrac{a^2-ab}{a-b}=\dfrac{a(2-b)}{a-b}$.

Q. 11 : $\dfrac{3b-3a}{a-b}=\dfrac{3b}{a}-\dfrac{3a}{b}=\dfrac{3b^2-3a^2}{ab}$.

Q. 12 : $a+\dfrac{1}{b}=\dfrac{ab+1}{1+b}$.

Q. 14 : $\dfrac{1}{a}-b=\dfrac{1-ab}{a-1}$.

Q. 15 : $\dfrac{a}{b}+\dfrac{3}{b}=\dfrac{a+3}{2b}$; also $\dfrac{3a}{2b}$; also $\dfrac{ab+3b}{b^2}=a+3$ by Partial Cancelling ; and also $\dfrac{ab+3b}{b}=a+3$ (See Q. 22 comment).

Q. 18 : $\dfrac{y}{a+b}-\dfrac{3}{a+b}=\dfrac{y(a+b)-3(a+b)}{(a+b)(a+b)}$ by routine process without thought, then either left in that form or brought by Partial Cancelling to $y-3$.

Q. 19 : $\dfrac{x}{a-b}-\dfrac{y}{b-a}=\dfrac{x(b-a)-y(a-b)}{(a-b)(b-a)}=x-y$.

Q. 20 : $\dfrac{5}{2b}-\dfrac{3}{b}=\dfrac{5-3b}{2b}$;

also $\dfrac{2}{b}$, presumably from $\dfrac{5-3}{2b-b}$;

also $\dfrac{7}{b}$, from $\dfrac{10}{b}-\dfrac{3}{b}$?

Q. 22 : $\dfrac{a}{2b}+\dfrac{c}{b}=\dfrac{a+bc}{2b}$;

also $\dfrac{ab+2cb}{2b}$ arising from a routine of cross-multiplying before deciding the denominator.

Q. 23 : $\dfrac{a}{2b}-\dfrac{c}{b}=\dfrac{-ab-2bc}{-2b^2}$, inserting the $-$ in both numerator and denominator, $\dfrac{-c}{-b}$;

also $\dfrac{a-bc}{2b}$.

Q. 24 : $\frac{3}{a+b}+\frac{7}{4(a+b)}=\frac{10}{4(a+b)}=\frac{5}{2(a+b)}$;

also $\frac{3\times\text{“}3\text{”}+7}{4(a+b)}=\frac{16}{4(a+b)}=\frac{4}{a+b}$.

Q. 26 : $\frac{a+3}{a+b}+\frac{c+4}{2(a+b)}=\frac{c+10}{b}$. Via $\frac{2a+6+c+4}{2a+b}=\frac{c+10}{b}$?

Q. 28 : $\frac{1}{b}-\frac{3}{d}=\frac{d-3b}{-bd}$ (cf. Q. 23).

Q. 30 : $\frac{a}{b}+\frac{c}{d}=\frac{a+c}{b+d}$,

Q. 31 : $\frac{a}{b}+\frac{2}{b+c}=\frac{2b+a(b+c)}{b(b+c)}=\frac{2b+a}{b}$.

Q. 32 : $\frac{1}{a+b}-\frac{5}{c+d}=\frac{(c+d)-5(a+b)}{(a+b)(c+d)}=-5$.

Q. 33 : $\frac{a}{a+b}+\frac{e}{c+d}=\frac{ae}{(a+b)+(c+d)}$.

Q. 34 : $\frac{2a}{a+b}+\frac{4c}{a+4c}=\frac{a^2+6ac+2bc}{a(b+4c)}$, this denominator then having been crossed out and replaced by $a+b+a+4c$. The first attempt seems to arise from $a\times b+a\times 4c$. The numerator has lost a factor 2.

Q. 35 : $\frac{1}{a}+\frac{3}{a+b}=\frac{1(a)+3(a+b)}{a(a+b)}$.

Q. 39 : $\frac{a}{b}\times\frac{c}{d}=cb\times ad$. (A confusion of $+$, $\times$, and omitting the denominator, or a complete misunderstanding?)

Q. 44 : $\frac{ab}{ac}\div bc=a^2b^2c^2$.

Q. 46 : $\frac{5a}{a+b}\times\frac{c}{a}=5\times c(a+b)$;

also $\frac{c(a+b)}{5a^2}$ and $5a^2c(a+b)$.

Q. 52 : $\frac{abc}{a+b+c}\times\frac{a+b}{a+c}=\frac{abc(a+b)(a+b+c)(a+c)}{(a+b+c)(a+c)}=abc(a+b)$,

probably from a confusion of $+$, $\times$, giving $\frac{abc}{a+b+c}+\frac{a+b}{a+c}$.

Results

The percentage of scores in various significant ranges were

Score	0–9	10–14	15–19	20–52
Percentage of papers	18	18	13	51

Of the 1053 papers marked, 539 (51·2%) scored 20 or more and were therefore analysed. The average score on these was 32·7 and the highest 51. Table II indicates the frequency of errors in each category for each question. The number of errors in each category is given as a percentage of 539, corrected to the nearest integer, as any show of greater precision would be spurious from

the uncertainty of the material. For example, in Q. 26 there were 64 failures to add together the pure numbers in the numerator, so in column *M* there is a percentage $\frac{64}{539} \times 100 = 11{\cdot}87$, rounded off to 12. An asterisk * indicates a positive entry less than 0·5% (which arises from either one or two occurrences).

The figures in row T indicate the frequency of each category of error, as a percentage of 539. For example, the total 384 in row T, column C, indicates that on the average in 100 papers cancelling errors appear in 384 answers. These figures do not always quite agree with the column-totals, because of the rounding off process.

The figures in column S indicate the percentage number of errors for each question, e.g. that for Q. 31 there are 63 errors on the average in 100 papers. Again, these figures do not always quite agree with the row-totals, for the reason just given. The numbers in this column must be interpreted with two considerations in mind acting in opposite directions. Firstly, that one wrong answer may yield more than one type of error. Secondly, that questions which have not been attempted do not yield an error-entry although they must be counted as wrong. A column giving the percentage of times that each question was attempted would help the interpretation of these numbers but would contain its own uncertainties, e.g. whether a question had been seriously attempted or had received a hasty last-minute guess. Instead of this, therefore, the total number of correct answers to each question is given in column T, and these two adjacent columns S and T together give a sounder indication of performance on each question. Column S alone suggests that Q. 31 was answered less well than either Q. 51 or 52, but T shows that Q. 31 was answered correctly more often than Q. 51 or 52, (48% compared with 27% and 34%). The discrepancy is mainly owing to the number of times that Q. 51, 52 were not attempted. Q. 19 and 49 may be compared in the same way.

Reliability of Results

There are considerable differences in the figures from school to school, to be expected from the variation in the abilities of pupils and teachers, and other factors. To illustrate this the last two columns Y, Z and bottom two rows Y, Z are given for sub-groups Y, Z of the papers. Group Y contains 308 out of 617 from 6 schools, analysed by one of the collaborators, and Z contains the remaining 231 out of 436 from 3 schools, analysed by the other collaborator. If, for a given question, the figures in columns Y, Z are in fair agreement, then the weighted mean (already given in column T) may reasonably be taken as a norm. For Q. 11, 26, 31, 35, 47, 48, 51, 52 the discrepancy is certainly too great to justify our taking the numbers in column T as norms.

The rows Y, Z give the percentage occurrence of each type of error within these sub-groups. The discrepancies between these pairs of numbers arise not only from differences between the schools but also from any differences in judgment between the two investigators. Only columns K, L and perhaps M show agreement, giving the possibility of a norm.

Comments from the Standpoint of the Mathematics Teacher

The most striking feature is the serious misconceptions that are revealed even in papers whose score is above the average. Do algebra lessons (we cannot say " the study of algebra ") achieve anything of value with the non-mathematician if so little is understood? Could this failure be avoided by better teaching, or is it inevitable with the raw material of mental abilities with which the pupils are endowed?

The significance of the failure is not limited to algebra, but extends also to the realm of arithmetic in the confusion of multiplication and addition, errors in denominators and numerators, and inversion. The confusion of multiplication and addition may be partly due to the resemblance between the signs + and × and partly to the difference in convention in arithmetic and algebra about

the meaning of juxtaposition, but it also suggests that teaching time needs to be devoted to an examination of the differences between the two processes, as exemplified in $3 \times (2 + 5)$ and $3 + (2 \times 5)$. Techniques are probably too much taught without reference to the underlying nature of the processes. An understanding of this difference between multiplication and addition would prevent "partial cancelling" from occurring, and a fuller discussion of the method of a common denominator in fractions would reduce the numerator and denominator types of errors. We cannot perhaps expect this understanding to be fully reached at the stage when arithmetical fractions are usually taught, but meeting the topic in algebra a few years later would provide the opportunity for a re-discussion and consolidation of the principles involved. The three most frequently occurring categories of error, A, B, C would thus be largely eliminated.

TABLE 1

FULL NAME

AGE YEARS MONTHS

Below you will find a number of simple examples in algebraic fractions. When you are told to begin start at No. 1 and work through them all. Bring each to a single fraction in its lowest terms. If any cannot be brought to lower terms say so. Write the answers straight down if you can. Rough work can be done on the paper provided.

You will find that not all the easiest examples are at the beginning so do not waste time over any you cannot do.

If there are any questions ask them now. You will not be allowed to ask them once you have begun.

Addition and Subtraction

(1) $\frac{2a}{2} =$

(2) $\frac{4}{2a} =$

(3) $\frac{pa}{pb} =$

(4) $\frac{a^2}{ab} =$

(5) $\frac{6}{6a + 6b} =$

(6) $\frac{3}{6a + b} =$

(7) $\frac{3a - b}{b} =$

(8) $\frac{3a + 3b}{a + b} =$

(9) $\frac{ka + kb}{a + b} =$

(10) $\frac{a^2 - ab}{a - b} =$

(11) $\frac{3b - 3a}{a - b} =$

(12) $a + \frac{1}{b} =$

(13) $1 + \frac{a}{b} =$

(14) $\frac{1}{a} - b =$

(15) $\frac{a}{b} + \frac{3}{b} =$

(16) $\frac{1}{a + b} + \frac{3}{a + b} =$

(17) $\frac{x}{a+b}+\frac{y}{a+b}=$

(18) $\frac{y}{a+b}-\frac{3}{a+b}=$

(19) $\frac{x}{a-b}-\frac{y}{b-a}=$

(20) $\frac{5}{2b}-\frac{3}{b}=$

(21) $\frac{a}{3}+\frac{c}{9}=$

(22) $\frac{a}{2b}+\frac{c}{b}=$

(23) $\frac{a}{2b}-\frac{c}{b}=$

(24) $\frac{3}{a+b}+\frac{7}{4(a+b)}=$

(25) $\frac{3x}{a+b}-\frac{2y}{4(a+b)}=$

(26) $\frac{a+3}{a+b}+\frac{c+4}{2(a+b)}=$

(27) $\frac{1}{b}+\frac{1}{d}=$

(28) $\frac{1}{b}-\frac{3}{d}=$

(29) $\frac{1}{b}+\frac{c}{d}=$

(30) $\frac{a}{b}+\frac{c}{d}=$

(31) $\frac{a}{b}+\frac{2}{b+c}=$

(32) $\frac{1}{a+b}-\frac{5}{c+d}=$

(33) $\frac{a}{a+b}+\frac{e}{c+d}=$

(34) $\frac{2a}{a+b}+\frac{4c}{a+4c}=$

(35) $\frac{1}{a}+\frac{3}{a+b}=$

(36) $\frac{a}{a+b}+\frac{b}{a+b}=$

(37) $\frac{a}{a+b}-\frac{c}{a+c}=$

Multiplication and Division

(38) $a\times\frac{1}{c}=$

(39) $\frac{a}{b}\times\frac{c}{d}=$

(40) $\frac{a^2}{b}\times\frac{c}{a}=$

(41) $\frac{a}{b}\times\frac{b}{a}=$

(42) $a\div\frac{a}{b}=$

(43) $\frac{a^2}{b}\div\frac{b}{a}=$

(44) $\frac{ab}{ac}\div bc=$

(45) $a^3\div\frac{1}{a^2}=$

(46) $\frac{5a}{a+b}\times\frac{c}{a}=$

(47) $\frac{3a+b}{a}\times\frac{b}{a+b}=$

(48) $\frac{a^2+b}{a+3b}\times\frac{2a+6b}{a^2+3}=$

(49) $\frac{a^3+b^2}{2b-a}\times\frac{3a-6b}{a^2+b}=$

(50) $\frac{3ab}{a^3+ab}\times\frac{a}{5b}=$

(51) $\frac{2a}{2a+b}\times\frac{2ab+b^2}{6a^2}=$

(52) $\frac{abc}{a+b+c}\times\frac{a+b}{a+c}=$

TABLE 2

Question	% of Papers showing Errors of Type A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	Aggregate Error % S	% of Correct Answers for each question: Whole T	Sub-group Y	Sub-group Z
1	–	–	–	–	–	–	2	–	–	–	–	•	–	–	–	2	98	98	98
2	–	–	•	–	2	–	3	–	–	–	–	–	–	2	–	7	93	92	94
3	–	–	•	–	–	–	3	–	–	–	–	–	–	2	–	5	95	93	98
4	–	–	1	–	–	–	4	–	–	–	–	–	–	2	•	6	94	95	93
5	–	–	20	–	3	–	6	–	–	–	1	1	–	3	–	34	71	71	72
6	–	–	32	–	2	–	•	–	•	–	1	1	–	2	–	38	62	68	54
7	–	–	22	–	–	–	–	–	•	–	•	–	–	2	–	24	69	74	63
8	–	–	17	3	–	–	–	4	•	–	•	1	–	2	–	27	73	75	70
9	–	–	23	–	–	–	–	4	•	–	•	1	–	2	–	30	70	69	70
10	–	–	24	•	2	–	–	7	•	–	–	–	–	3	–	34	67	68	66
11	–	–	12	2	–	–	–	2	–	40	–	1	–	3	•	60	39	31	48
12	1	3	1	–	•	51	–	–	–	–	5	6	–	1	•	68	39	40	38
13	1	2	•	–	–	45	–	–	–	–	5	9	–	•	–	62	45	45	46
14	1	3	•	–	•	50	–	–	–	–	5	3	–	•	•	63	42	42	42
15	4	6	1	–	–	–	•	–	–	–	3	7	–	•	•	21	81	80	82
16	2	3	1	–	•	–	–	1	–	–	2	1	10	1	•	21	79	80	78
17	2	3	•	1	•	–	–	1	•	–	2	2	–	•	1	11	91	92	89
18	1	2	1	–	•	–	–	•	1	–	3	1	–	1	1	10	90	90	90
19	2	5	2	–	•	–	–	–	1	60	3	1	–	1	1	75	26	23	30
20	7	11	•	–	•	–	3	–	–	–	2	•	7	1	3	33	67	71	61
21	3	5	4	–	–	–	3	–	•	–	1	•	–	1	1	18	84	87	80
22	5	10	3	–	–	–	2	–	–	–	1	•	–	•	•	21	81	82	79
23	5	11	2	–	–	–	2	–	–	–	1	–	–	•	•	23	80	81	79
24	4	7	2	–	–	–	–	1	•	–	3	•	6	1	1	25	74	76	72
25	5	7	5	–	–	–	49	1	1	–	3	•	–	2	1	72	22	20	25
26	5	7	10	–	–	–	–	1	4	–	2	•	12	1	1	42	53	61	43
27	8	6	1	–	–	–	1	–	–	–	2	1	–	2	•	20	79	80	79
28	7	8	1	–	–	–	•	–	–	–	2	•	–	2	2	22	75	76	74
29	7	8	2	–	–	–	–	–	–	–	2	•	–	1	1	21	79	81	76
30	6	6	3	–	–	–	•	–	–	–	2	•	–	1	•	19	81	85	76
31	28	20	6	–	•	–	–	–	3	–	3	2	–	1	1	63	48	55	39
32	9	7	9	–	–	–	–	–	10	–	3	1	–	1	2	41	52	56	47
33	7	6	10	–	•	–	–	–	1	–	4	1	–	1	2	32	61	64	56
34	17	12	9	–	–	–	–	•	3	–	4	1	16	2	3	67	27	32	21
35	22	14	7	–	–	–	–	–	3	–	3	1	13	2	1	64	36	43	25
36	2	4	5	•	–	–	–	12	•	–	3	•	–	1	–	29	64	69	58
37	15	14	8	–	–	–	–	–	8	–	4	•	15	3	2	69	23	24	22
38	–	–	•	–	2	–	–	–	–	–	–	1	–	3	•	6	88	90	86
39	–	–	•	–	1	–	–	–	–	–	–	2	–	2	•	6	89	91	87
40	–	•	–	–	1	–	11	–	–	–	–	2	–	2	1	18	81	80	82
41	–	–	12	–	1	–	1	–	–	–	–	2	–	2	•	18	86	87	84
42	–	–	1	•	10	–	4	–	–	–	–	1	–	1	•	18	80	79	81
43	–	–	•	–	11	–	•	–	–	–	–	1	–	3	1	17	79	79	80
44	–	–	–	–	11	–	29	–	–	–	–	2	–	6	•	48	44	44	43
45	–	–	–	–	6	–	–	–	–	–	–	1	–	4	–	11	82	83	80
46	–	–	2	3	2	–	10	–	2	–	–	3	–	2	1	25	66	66	65
47	–	–	25	1	1	–	–	–	17	–	–	3	–	3	•	51	37	44	29
48	–	–	22	1	–	–	–	12	5	–	–	2	–	4	2	48	37	44	29
49	–	–	27	•	–	–	–	3	8	29	•	1	–	4	2	74	8	9	6
50	–	–	17	–	–	–	24	–	3	–	–	1	–	2	3	52	23	26	18
51	–	–	18	–	1	–	11	13	1	–	–	1	–	3	3	51	27	34	18
52	•	–	15	–	–	–	–	–	7	–	•	2	–	2	1	27	34	42	23
																1746	3274	3368	3148
% Occurrence of each type of Error. Whole T	172		384	12	54	147	169	62	79	129	75	67	78	90	39				
% Occurrence of each type of Error. Sub-Gp. Y	146		295	17	68	106	200	73	114	170	74	69	86	123	50				
% Occurrence of each type of Error. Sub-Gp. Z	207		505	6	35	202	126	47	81	75	75	67	74	89	23				

• Indicates a positive entry less than 0·5.

THE SIMPLE PENDULUM AT ANY AMPLITUDE

By P. J. Bulman

Through the initiative of the senior mathematics master, Mr. R. D. Knight, a simple Bush Differential Analyser, was built almost entirely from "Meccano" parts at Kings School, Worcester in the last year.

In order both to test the machine and to obtain information I recently investigated the motion of a particular simple pendulum by means of the analyser.

To obtain an algebraic solution of the equation of motion for a simple pendulum

$$\ddot{\theta} + N \sin \theta = 0$$

it must be approximated to

$$\ddot{\theta} + N\theta = 0$$

which is true only for small oscillations.

Using the differential analyser, however, the motion can be solved rigorously, accuracy being dependent only on the machine. The solution thus obtained is not algebraic but geometric and it is a distortion of the pure sine curve met in simple harmonic motion. It is impossible to deduce an equation for this curve in terms of elementary functions.

This fact, however, is not the handicap it may seem, as the period and amplitude of the motion are readily ascertainable from the curve, which is drawn automatically by the machine, as are the angular velocity and corresponding displacement at any instant. Since this was essentially a mechanical and not an algebraic investigation nothing else was required.

The arrangement of the machine was as follows :

The rotation of shaft a in the diagram represents $\ddot{\theta}$ and displaces turntable II. The output from this table (shaft b) represents $\dot{\theta}$ and displaces turntable I. The output from this (shaft c) represents θ and displaces the vertical axis of the output table and the horizontal axis of the input table. On the input table is placed a sine curve such that when this graph is followed manually the rotation of shaft d represents $3{\cdot}81 \sin \theta$.

The rotations of shafts a and d are linked in the gears e such that the equation

$$\ddot{\theta} = -3{\cdot}81 \sin \theta$$

is formed.

The horizontal (time) axis of the output table is displaced and the turntables are rotated by an electric motor running at an arbitrary speed.

The initial setting of turntable I gives the initial value of $\dot{\theta}$ and hence, by energy considerations, the amplitude. The settings of turntable II and of the input table must be zero coincidentally and such that the pencil on the output table is on the time axis when this is so.

As "Meccano" gears of ratios 1 : 1 and 3 : 1 were the only ones available, the number of possible pendulums was severely restricted. I chose one 8 ft. 4·5 ins. long as this entailed least gearing up, which is undesirable in any machine.

The corresponding S.H.M. :

$$\ddot{\theta} + 3{\cdot}81\theta = 0$$

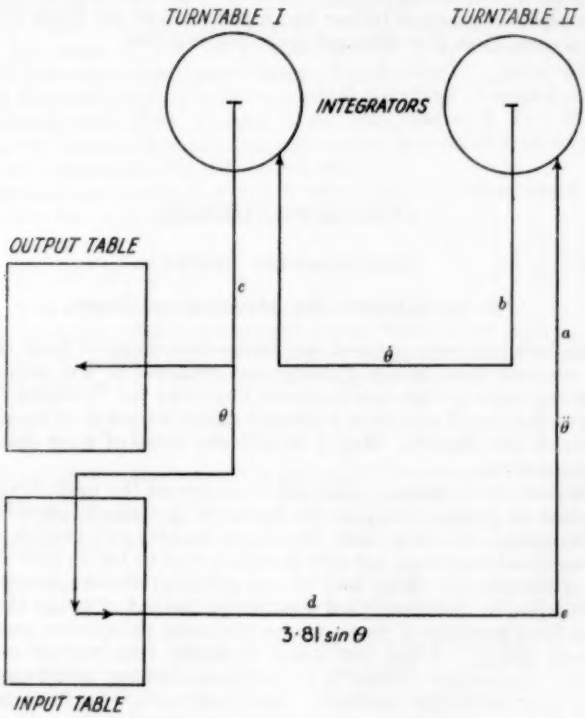
has a period of 3·22 seconds.

The equation of motion of the pendulum is ;

$$\ddot{\theta} + 3{\cdot}81 \sin \theta = 0.$$

A series of seven runs was then carried out and a distorted sine curve obtained for each. The most obvious effects of this distortion were a marked

steepening of the slope at points of inflection, together with an increase in the " bluntness " at maxima and minima.



The amplitude and period of the swing of the pendulum bear a linear relationship with the same quantities on the curve produced by the machine and hence they were obtained and tabulated below :

Run	*Amp.*	*Period*
1.	25·8°	3·3 secs.
2.	51·7°	3·46 secs.
3.	71·5°	3·68 secs.
4.	96·5°	3·91 secs.
5.	105·2°	4·22 secs.
6.	123·2°	4·67 secs.
7.	155·0°	5·94 secs.

Due to the inherent inaccuracy of the machine, which was caused mainly by backlash in the gears, it proved impossible to obtain an amplitude less than 25° or greater than 155° with any reliability; certain deductions, however, can be drawn outside these limits.

As the amplitude is reduced it will approximate more and more closely to the S.H.M. and have a minimum period of 3·22 secs. when $\theta_{max} = 0$. When the

amplitude is increased to 180° the period will tend to infinity, since the system no longer behaves as a simple pendulum but rotates in a vertical circle.

From this it follows that if a graph of θ_{max} against period is drawn, it will be symmetrical about the line $\theta_{max} = 0$, and it will have asymptotes at $\pm 180°$ Apart from this the graph is rather like a parabola of the form $X^2 = Y - 3{\cdot}22$ except, of course, that this does not have asymptotes.

P. J. B.

CORRESPONDENCE

THE GEOMETRY REPORT

To the Editor of the *Mathematical Gazette*.

DEAR SIR,—It has been pointed out to me that some of your readers have expressed surprise that in my Presidential Address to the Association last April I did not refer to the Association's *Report on the Teaching of Geometry*, and that the absence of any such reference might be taken to mean that I did not approve of the Report. May I be allowed some of your space to refute any such suggestion.

As the Report (p. 2) states, "The point of view of the main body of mathematicians, and of groups, such as the lecturers in a single university, is continually changing." Coming from one of our ancient universities which has a strong geometrical tradition, my sole intention was to try to give my audience some idea of the current views held by one group of contemporary geometers, and in particular to point out that geometry, instead of being the unwanted child it was for a number of years, is now the most prosperous member of the mathematical family. I did not mean to imply that recent developments called for any immediate change in school mathematics, and I merely wished to share the good news that geometry once more occupies a fundamental place in mathematical science with my audience, most of whom had studied geometry at a university.

I have re-read the Report, and find nothing in it which I wish to criticise; indeed, I am full of admiration for the way in which the Report reconciles the opposing needs of dealing on the one hand with the unwilling school boy and, on the other, of providing a sound foundation in the principles of geometry for the more gifted pupils. As I said in my Address, my experience does not qualify me to give advice to those who have laboured long in the training of school children. If I make any plea, it is that in sixth forms, and elsewhere, the emphasis should be on a few basic principles, and riders should only be used to illustrate these principles and to promote enjoyment of the subject, never as a penance.

Yours faithfully,

28 Barrow Road, Cambridge. W. V. D. HODGE.

FAREY SERIES AND STERN SERIES

By C. G. Paradine

1. The Farey series F_n consists of all the proper fractions, in their lowest terms and in order of magnitude from 0/1 to 1/1, whose denominators do not exceed n. The series F_n may be formed from F_{n-1} by inserting the mediant $(a+c)/(b+d)$ between two adjacent terms a/b and c/d of F_{n-1}, provided $b+d \leqslant n$. The process requires inspection of terms and rewriting of earlier series. Professor Neville showed (*Proc. London Math. Soc.*, Series 2, Vol. 51), how the order n may be increased approximately in the ratio 3 to 2 at one time, but inspection and repetition will still be necessary.

If all mediants are inserted without restriction of denominator we obtain the simpler but less useful series

$$\frac{0}{1}, \frac{1}{1}; \frac{0}{1}, \frac{1}{2}, \frac{1}{1}; \frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}; \frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1};$$

and so on. The denominators are now the Stern series $S_1(1, 1)$, $S_2(1, 1)$,... while the numerators, after the first set, are the Stern series $S_1(0, 1)$, $S_2(0, 1)$,... joined to $S_1(1, 1)$, $S_2(1, 1)$,... If the series of fractions are similarly joined after removal of 0/1, there is formed the single series

$$\frac{0}{1}; \frac{1}{1}; \frac{1}{2}, \frac{1}{1}; \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}; \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1}; \quad \text{.....(A)}$$

To form the denominators of each new group, add numerator to denominator in the preceding group and complete the denominators of the new group by symmetry. The numerators of the new group are the set of all earlier denominators. Keeping only the odd terms of (A), we have the series

$$\frac{0}{1}; \frac{1}{2}; \frac{1}{3}, \frac{2}{3}; \frac{1}{4}, \frac{2}{5}, \frac{2}{5}, \frac{3}{4}; \frac{1}{5}, \frac{2}{7}, \frac{3}{8}, \frac{3}{7}, \frac{4}{7}, \frac{5}{8}; \ldots$$

which, after the first two terms, is formed by the same rule as (A). Each fraction now appears once only, but to place them in order of magnitude we must interlace the first two with the second two, then those four with the next four, and so on. The generation of this series or of series (A) or of the denominators of (A) could presumably be easily coded for computing machines.

If (A) is stopped at 2^n terms, then, with the fraction 0/1, the last group forms a series A_n, say, of $2^{n-1}+1$ fractions in order of magnitude. The corresponding Farey series F_n has about $3n^2/\pi^2$ terms. As n increases it is soon not practical to construct F_n from A_n by cancelling those fractions whose denominators exceed n. A practical alternative might be to derive from A_n the series F_m having about the same number of terms, so that $3m^2/\pi^2 \simeq 2^{n-1}$. After cancelling fractions of A_n with denominators greater than m, we must insert about the same number of fractions with denominators r, where $n < r \leqslant m$. These will be required in the neighbourhood of the smaller denominators, the largest group occurring between 0/1 and $1/n$, with a complementary group between $(n-1)/n$ and 1/1. As an experiment, F_{41} was constructed from A_{10}. Of the 513 terms of the latter, 220 had to be cancelled, and 238 new terms had to be inserted.

2. A simple formula for the nth fraction of (A) can be obtained. If $f(n)$ denotes the nth denominator, then $f(2m) = f(m)$ and so $f(2^r m) = f(m)$. We consider the case of n odd and express n in the binary scale, so that its digits consist of alternate sequences of the integers 1 and 0. Let $a_1, a_2, \ldots a_{2m+1}$ be the number of digits in each sequence. (For instance, if $n = 990979$ in the decimal scale, $n = 11110001111100000011$ in the binary scale and $a_1 = 4$, $a_2 = 3$, $a_3 = 5$, $a_4 = 6$, $a_5 = 2$.) Using the relations

$$f(2^r m) = f(m), \quad f(n) = f(n+1) + f(n-1) \quad (n \text{ odd})$$

we have

$$\begin{aligned} f(n) &= f(a_1, a_2, \ldots, a_{2m+1}) \\ &= f(n+1) + f(a_1, a_2, \ldots, a_{2m+1} - 1) \\ &= f(n+1) + f(a_1, a_2, \ldots, a_{2m} - 1, 1) + f(a_1, a_2, \ldots, a_{2m+1} - 2) \\ &= f(n+1) + f(n+1) + f(a_1, a_2, \ldots, a_{2m+1} - 2) \end{aligned}$$

and by repeated application

$$\begin{aligned} f(n) &= a_{2m+1} f(n+1) + f(a_1, a_2, \ldots, a_{2m-1}) \\ &= a_{2m+1} f(n+1) + f(n'), \text{ say.} \end{aligned} \quad \ldots\ldots(1)$$

(If $a_{2m} = 1$, the term $f(a_1, a_2, \ldots, a_{2m} - 1, 1)$ reduces to $f(a_1, a_2, \ldots, a_{2m-1} + 1)$ but is still $f(n+1)$).

Also
$$\begin{aligned} f(n+1) &= f(a_1, a_2, \ldots, a_{2m} - 1, 1) \\ &= f(a_1, a_2, \ldots, a_{2m} - 2, 1) + f(n') \end{aligned}$$

and by repeated application

$$\begin{aligned} f(n+1) &= f(a_1, a_2, \ldots, a_{2m-1} + 1) + (a_{2m} - 1) f(n') \\ &= f(2n'+1) + (a_{2m} - 1) f(n') \\ &= f(n'+1) + a_{2m} f(n'). \end{aligned} \quad \ldots\ldots(2)$$

Substituting in (1),

$$f(n) = (a_{2m} a_{2m+1} + 1) f(n') + a_{2m+1} f(n'+1) \quad \ldots\ldots(3)$$

From (2) and (3) we can now prove, by induction on m, that $f(n)/f(n+1)$ is equal to the continued fraction

$$a_{2m+1} + \frac{1}{a_{2m}+} \; \frac{1}{a_{2m-1}+} \cdots \frac{1}{a_1}.$$

For if

$$a_{2m-1} + \frac{1}{a_{2m-2}+} \cdots \frac{1}{a_1} = \frac{f(n')}{f(n'+1)},$$

then

$$\begin{aligned} a_{2m+1} + \frac{1}{a_{2m}+} \cdots \frac{1}{a_1} &= \frac{(a_{2m} a_{2m+1} + 1) f(n') + a_{2m+1} f(n'+1)}{a_{2m} f(n') + f(n'+1)} \\ &= \frac{f(n)}{f(n+1)}. \end{aligned}$$

Now if $n' = (a_1) = 2^{a_1} - 1$, $f(n') = a_1$ and $f(n'+1) = f(2^{a_1}) = 1$. The result is therefore true when $m = 1$ and so by induction for all values of m.

So far we have formulae for two consecutive denominators of (A). In the square bracket notation used by Davenport, *The Higher Arithmetic*, write*

$$\begin{aligned} f(n) &= [a_{2m+1}, a_{2m}, \ldots, a_1] \\ f(n+1) &= [a_{2m}, \ldots, a_1]. \end{aligned}$$

* Hardy and Wright, *Theory of Numbers*, use the square brackets for the continued fraction.

The corresponding consecutive numerators are earlier denominators, namely $f(m')$ and $f(m'+1)$, where $m' = (a_1 - 1, a_2, \ldots, a_{2m+1})$ in the binary scale. The nth fraction of (A) is

$$\frac{f(m')}{f(n)} = \frac{[a_{2m+1}, a_{2m}, \ldots, a_1 - 1]}{[a_{2m+1}, a_{2m}, \ldots, a_1]}.$$

By virtue of Euler's rules for the evaluation of the square bracket functions, the elements may be reversed, giving

$$\begin{aligned}\frac{f(m')}{f(n)} &= \frac{[a_1 - 1, a_2, \ldots, a_{2m+1}]}{[a_1, a_2, \ldots, a_{2m+1}]} \\ &= \frac{[a_1, a_2, \ldots, a_{2m+1}] - [a_2, a_3, \ldots, a_{2m+1}]}{[a_1, a_2, \ldots, a_{2m+1}]} \\ &= (p - q)/p\end{aligned}$$

where

$$\frac{p}{q} = a_1 + \frac{1}{a_2 +} \; \frac{1}{a_3 +} \cdots \frac{1}{a_{2m+1}}.$$

The nth fraction of (A) is

$$1 - \frac{q}{p} = 1 - 1 \Big/ \left\{ a_1 + \frac{1}{a_2 +} \cdots \frac{1}{a_{2m+1}} \right\}.$$

The $(n+1)$th fraction is

$$\begin{aligned}\frac{f(m'+1)}{f(n+1)} &= \frac{[a_{2m}, a_{2m-1}, \ldots, a_1 - 1]}{[a_{2m}, a_{2m-1}, \ldots, a_1]} \\ &= \frac{[a_1 - 1, a_2, \ldots, a_{2m}]}{[a_1, a_2, \ldots, a_{2m}]} = 1 - 1 \Big/ \left\{ a_1 + \frac{1}{a_2 +} \cdots \frac{1}{a_{2m}} \right\}.\end{aligned}$$

In the example, $n = 990979 = (4, 3, 5, 6, 2)$, we find

$$4 + \frac{1}{3 +} \; \frac{1}{5 +} \; \frac{1}{6 +} \; \frac{1}{2} = \frac{923}{214}$$

while the penultimate convergent is 427/99. The nth fraction is

$$1 - 214/923 = 709/923$$

and the $(n+1)$th is $1 - 99/427 = 328/427$. These two fractions are neighbours in the Farey series of orders from 923 up to $(923 + 426)$.

The place of first appearance in (A) of a given fraction h/k is found by expressing $k/(k-h)$ as a continued fraction with an odd number of terms or partial quotients $a_1, a_2, \ldots, a_{2m+1}$. Then h/k is the nth term where

$$n = (a_1, a_2, \ldots, a_{2m+1})$$

in the binary scale. Since $k = f(n) = [a_1, a_2, \ldots, a_{2m+1}]$, a given denominator k will occur in an odd place as often as it can be made the numerator of a simple continued fraction with an odd number of partial quotients. This is otherwise evident since it can be so expressed after division by any integer less than and prime to k.

C. G. P.

UNIFIED MATHEMATICS OR MATHEMATICS AS A UNITY?

By N. E. W. Chapman

In a recent book review in the *Mathematical Gazette* it was stated that the idea of a unified syllabus in mathematics had proved less helpful than had been hoped, and it was implied that many teachers were abandoning the idea of a unified syllabus, or were compromising with a system of partial unification. This was only one of a number of articles and reviews in various periodicals, all noticeably tepid in their enthusiasm for the new ways, which must cause some dismay to all those who have hoped for much good from this newest phase in the development of mathematical teaching. It is, of course, hardly surprising that so momentous a step should have given rise to a certain amount of controversy : the dethronement of Euclid was the cause of a great deal of fierce argument as well as of the birth of the Mathematical Association, so that the establishment of a unified syllabus, another and greater step along the same road, must be expected to receive an equally mixed reception. Nevertheless, if the unified syllabus has indeed been found unsatisfactory by some teachers, it would be well to try to find out what is wrong.

It has been said of the Christian religion that it has not been tried and found wanting, but has been found difficult and not tried : if the comparison is not irreverent, the same thing might almost be said of the unified syllabus in mathematics. It is true that a large number of text-books have been published, and continue to be published, which are avowed by their authors to be based on the idea of unification. Many of these are excellent books, or at least they contain chapters which, taken individually, are excellent, and they give very good treatment of many individual topics ; but hardly one of them makes any attempt at a true unification of mathematics. In nearly all of them we find a mixture of chapters on arithmetic, chapters on algebra and chapters on geometry (sometimes lifted bodily from the appropriate " separate branch " books by the same writer), each chapter bearing little or no relation to what comes before it or to what goes after it. This is not unification but simply mixing up, and indeed the separate topics, like oil and water shaken up together, stubbornly refuse to be mixed at all. Apart from the compactness and the saving in cost brought about by having one book instead of three or more, this arrangement brings no very great advantage to the teacher, who is normally quite capable of deciding for himself in what order the various topics are to be taken. Whatever the blessings of unification may be, they are not to be won by having arithmetic chapters, algebra chapters and geometry chapters in one book, and calling it all " Mathematics ".

The various branches of mathematics are taught in ways whose origins are so diverse that it is not surprising that they cannot be unified by a mere change of name. Arithmetic is a " unified syllabus " in itself, with " pure " and " applied " branches, the latter being again subdivided into the old commercial arithmetic (which still dominates the arithmetic papers in the old G.C.E. syllabus, even if such words as " alligation " are no longer heard) and mensuration, with a variety of miscellaneous topics. Algebra is less formal than it was, but still is largely dealt with in the attitude summarized by the chapter-heading, " problems on equations " (instead of equations as a means to solve problems). Geometry is, of course, still dominated by the persistent ghost of Euclid, although in these days few schoolchildren have ever heard his name : in spite of all the din of controversy, the resemblances between Euclid and most modern pre-unification text-books of elementary geometry are far more striking than the differences, with the emphasis on theorems each followed by a string of examples. Sometimes there is at the beginning of the book a grudging concession to Stage A ideas, out of deference to the Mathe-

matical Association report, but one can almost hear the sigh of relief when the author leaves this childishness and turns to *real* geometry.

It is only to be expected that these converging but different streams of learning cannot be unified by the mere change of a name, and that any attempt to unite them in such a way gives rise to disappointment. Yet few, surely, would wish to go back to the days—not very many years back in some schools and, it is whispered, still present to this very day in others—when Arithmetic and Algebra and Geometry were regarded as different subjects, not only examined in different examinations but treated as separate subjects in the time-table and taught by different teachers, with different places in mark-sheets and report forms. Those were the days when a problem set in, say, an algebra lesson, had to be solved by algebra and not by arithmetic, and trigonometrical ratios could never be used in a geometry lesson. Thus " Mathematics ", a set of artificial techniques, fought against common-sense instead of acting as a reinforcement and a systematization of it, and it can well be believed that the victims of teaching of this kind suffered from a very definite lowering of their ability to tackle problems of any slightly unusual nature. Moreover the whole body of education, ever since it lost the classics as its centre, has been seeking a unity of its own and the unity of mathematics is needed as a contribution to that larger unity.

If it is allowed that unified mathematics is desirable, but cannot be achieved by a mere mixing of chapters, the question then arises, how *is* it to be achieved? This question requires far more thought than it seems so far to have received. It seems that unification can best be accomplished *round* something, some definite centre, as a vapour condenses most readily around a particle of dust. This centre may be some interest outside the subject itself, an interest which will probably also serve as a unifying focus for other subjects in the curriculum (the topic of " railways " is suggested in the Modern School Report as an example of such a centre.) On the other hand, the centre of interest may be some probably more abstract idea within the subject of mathematics itself, and this may be the more suitable method for the Grammar School.

As just one example of what is meant by this kind of unity, one might consider the idea of building up the syllabus for the beginnings of secondary school work—in some ways the most difficult part—around the idea of *measurement*. From the beginning with geometrical themes—the accurate use of the ruler in drawing and measuring lengths—the work would naturally lead to fractions and decimals, as more precise means of expressing the results of measurement were needed. Scale drawing would lead to ratio and to calculations of the " decimalization of money " type, and the question of area would involve the need for formulae, which is the proper beginning of algebra. Each topic, once started, could if desired be pursued beyond the point needed for immediate use—for example, fractions would be applied to other things besides length—but all would be branches from the same trunk instead of separate and unconnected cuttings.

Even the construction of a connected syllabus of this kind, however, may not produce sufficiently deep consideration of the whole question of teaching mathematics. To approach the matter from the point of view of certain topics—fractions, simultaneous equations, angle properties of the circle and the rest—which must somehow be " brought in ", and welded into the unified course, is to approach it entirely from the wrong end. If mathematics is truly a unity it is so by its own very nature, and it does not need to have unification imposed from outside upon all its various branches. We should not begin by saying, " we must teach (say) simultaneous equations, so let us see what justification we can find for teaching them, and how we can weld them into a unified whole with other topics ", but rather by saying, " Why do we teach mathematics at all? And what methods, what topics, will enable us best to

fulfil the purpose we have thus declared? " In other words, what is needed is not unified mathematics in the sense of mathematics which has had unification imposed upon it, but mathematics which is a unity from its first conception.

An investigation on the lines suggested above might bring about a fairly drastic rearrangement of ideas, with some upsetting of familiar orders, some dropping of fairly well established old friends, and some bringing forward into the early part of the work of topics previously thought more advanced, or suitable only for mathematical specialists. This last has already happened, of course, in the case of trigonometry, and of calculus and a few other topics in the " alternative syllabus ".

It is suggested that the reasons for teaching mathematics might be taken to be somewhat as follows : first, because a certain facility in calculation is essential in everyday life ; secondly, because a much greater body of mathematical knowledge is useful in so wide a variety of occupations as to justify teaching it universally—even those pupils who do not ultimately undertake any of these occupations will at least gain a little insight into the point of view of those who do : thirdly, because the man or woman who has some knowledge of mathematics is likely to be a better citizen and to have a better understanding of the world than the one who has none—nearly every page of every newspaper contains at least one article involving mathematical ideas, the demand note for the rates is backed with calculations involving decimals at least, and there has been at least one poster campaign which played on the reader's ignorance of statistics ; fourthly, because mathematics is " one of the supreme achievements of the human spirit " and an essential element of our culture. The fifth reason is that mathematics gives mental training. The psychologists may sneer and murmur about " exploded theories of transfer of training ", but every teacher of mathematics knows how his subject, properly taught, can encourage precision of thought, a feeling for definition and the exact meanings of words, and a critical appreciation of logical reasoning.

If we now could decide what mathematics we must teach in order to fulfil the needs of each of these purposes, we should have a mathematical syllabus which would be truly a unity, for it would have been built up as a unity and not formed by forcing various separate parts together. The work of constructing such a syllabus is a task for a Mathematical Association committee continuing, but on a far deeper level, the work begun by the Jeffery report : the report of this new committee would be the most searching and profound yet written. Perhaps after its publication we could expect some truly " unified " text-books.

Cheshunt Grammar School N. E. W. CHAPMAN

1851. I wish in conclusion to express my humble thanks to Professor Littlewood, but for whose patient profanity this paper could never have become fit for publication.—S. Skewes, *Proc. London. Math. Soc.* (3) V, March 1955, p. 50. [Per Professor E. H. Neville.]

1852. HOW TO DRAW A STRAIGHT LINE.

When the Emperor Nicholas of Russia was asked to decide upon the route of the line between St. Petersburg and Moscow he contemptuously tossed aside the plans placed before him, ordered a map to be unrolled on the table, put his sword across the map, and drawing a straight line from one city to the other, regardless alike of ways and rights of property, flung his inexorable plan to the astounded surveyor, saying, " *Voilà votre chemin de fer.*"—John Pendleton, quoted in *The Railway Book : An Anthology edited by Stuart Legg* (*Rupert Hart-Davis*, 12*s.* 6*d.*). [Per Prof. H. D. Larsen.]

MATHEMATICAL NOTES

2585. *Inequalities involving upper and lower limits.*

Throughout this note $\{a_n\}$ and $\{b_n\}$ are bounded sequences of real numbers, Λ and λ are the upper and lower limits of $\{a_n\}$, Λ' and λ' are the upper and lower limits of $\{b_n\}$.

It is well-known that

$$\lambda+\lambda' \leqslant \underline{\lim_{n\to\infty}}\,(a_n+b_n) \leqslant \begin{matrix}\lambda+\Lambda'\\ \Lambda+\lambda'\end{matrix} \leqslant \overline{\lim_{n\to\infty}}\,(a_n+b_n) \leqslant \Lambda+\Lambda'.$$

In an (unsuccessful) attempt to find an alternative solution of a recent Tripos question I was led to the following inequalities:

$$\max(\lambda,\lambda') \leqslant \underline{\lim_{n\to\infty}}\,\max(a_n,b_n) \leqslant \begin{matrix}\max(\lambda,\Lambda')\\ \max(\Lambda,\lambda')\end{matrix} \leqslant \overline{\lim_{n\to\infty}}\,\max(a_n,b_n) \leqslant \max(\Lambda,\Lambda').$$

These results suggested the following theorem.

A necessary and sufficient condition that the inequalities

$$f(\lambda,\lambda') \leqslant \underline{\lim_{n\to\infty}}\,f(a_n,b_n) \leqslant \begin{matrix}f(\lambda,\Lambda')\\ f(\Lambda,\lambda')\end{matrix} \leqslant \overline{\lim_{n\to\infty}}\,f(a_n,b_n) \leqslant f(\Lambda,\Lambda') \qquad \ldots\ldots(1)$$

hold for all bounded sequences of real numbers $\{a_n\}$, $\{b_n\}$ *is that, for all real* x, y,

(i) $f(x,y)$ *be continuous, and*

(ii) $f(x,y)$ *be a non-decreasing function of* x *for fixed* y, *and a non-decreasing function of* y *for fixed* x.

(The above-mentioned results correspond to $f(x,y)=x+y$, $f(x,y)=\max(x,y)$ respectively.)

Write $$\lambda^* = \underline{\lim_{n\to\infty}}\,f(a_n,b_n), \quad \Lambda^* = \overline{\lim_{n\to\infty}}\,f(a_n,b_n).$$

Necessity. To show that $f(x,y)$ is continuous at a particular point (a,b) it is sufficient to show that if $\{(a_n,b_n)\}$ is any sequence of points tending to (a,b) then $f(a_n,b_n)$ tends to $f(a,b)$. Now if $\{(a_n,b_n)\}$ is such a sequence of points, then for the corresponding sequences of real numbers $\{a_n\}$, $\{b_n\}$

$$\Lambda=\lambda=a, \quad \Lambda'=\lambda'=b;$$

therefore, by (1),

$$f(a,b) \leqslant \lambda^* \leqslant f(a,b) \leqslant \Lambda^* \leqslant f(a,b);$$

hence $$\Lambda^*=\lambda^*=f(a,b),$$

and so $$f(a_n,b_n)\to f(a,b) \text{ as } n\to\infty.$$

To show that

$$f(x_1,y) \leqslant f(x_2,y) \text{ if } x_1<x_2$$

consider the sequences $\{a_n\}$, $\{b_n\}$ for which a_n is equal to x_1 or x_2 according as n is odd or even, and b_n is always equal to y. For these sequences

$$\lambda=x_1, \quad \Lambda=x_2, \quad \lambda'=\Lambda'=y,$$

and the result follows since, by (1), $f(\lambda,\lambda') \leqslant f(\Lambda,\lambda')$.

Sufficiency. Let ϵ be positive.

For all sufficiently large n

$$a_n>\lambda-\epsilon, \quad b_n>\lambda'-\epsilon;$$

therefore, by (ii),

$$f(\lambda-\epsilon,\ \lambda'-\epsilon)\leqslant f(a_n,\ b_n)$$

for all sufficiently large n, and so

$$f(\lambda-\epsilon,\ \lambda'-\epsilon)\leqslant\lambda^*.$$

On letting ϵ tend to zero it follows by (i) that

$$f(\lambda,\ \lambda')\leqslant\lambda^*.$$

Similarly it may be shown that $f(\Lambda,\ \Lambda')\geqslant\Lambda^*$.

For all sufficiently large n

$$a_n>\lambda-\epsilon,$$

and for an infinity of values of n

$$b_n>\Lambda'-\epsilon\ ;$$

therefore, by (ii),

$$f(\lambda-\epsilon,\ \Lambda'-\epsilon)\leqslant f(a_n,\ b_n)$$

for an infinity of values of n, and so

$$f(\lambda-\epsilon,\ \Lambda'-\epsilon)\leqslant\Lambda^*.$$

On letting ϵ tend to zero it follows by (i) that

$$f(\lambda,\ \Lambda')\leqslant\Lambda^*.$$

The remaining inequalities may be proved similarly.

L. E. CLARKE.

2586. *Multiplication of triangular matrices.*

We call a square matrix $A=(a_{pq})$ triangular if all its elements below the leading diagonal are zero, i.e. if $a_{pq}=0$ when $p>q$.

I. *The product of two triangular matrices is triangular.*

If $AB=C$, then we have (with the summation convention), $c_{pq}=a_{pr}b_{rq}$. Suppose $p>q$. When $r<p$, $a_{pr}=0$; when $r\geqslant p$, $b_{rq}=0$. Hence $c_{pq}=0$.

II. *The adjoint M of a triangular matrix A is triangular.*

Apart from sign, m_{pq} is equal to the minor of a_{qp}, i.e. to the determinant of the matrix X got from A by substituting 1 for a_{qp}, and 0 for every other element of its row or column. If $p>q$, a_{qp} is above the leading diagonal, and X is triangular, and its determinant is equal to the product of the elements of its leading diagonal. Two of these are zero, those namely replacing a_{qq} and a_{pp} ; hence $m_{pq}=0$.

Corollary 1. *If A is triangular and non-singular, A^{-1} is triangular.*

Corollary 2. *If A is triangular and non-singular, and AB or BA is triangular, B is triangular.*

For $$B=A^{-1}(AB)=(BA)A^{-1}.$$

M. F. EGAN.

2587. *A property of matrices.*

If D is the determinant of the $(n\times n)$ matrix A, A_1 the adjoint of A, and A_2 the adjoint of A_1, then

$$A_2=D^{n-2}A.$$

The following proof is due to Rev. R. E. Ingram :

If I is the unit matrix, $AA_1=DI$. Also, since Det $A_1=D^{n-1}$, we have $A_1A_2=D^{n-1}I$. Again,

$$(AA_1)A_2=A(A_1A_2),$$

whence the result follows.

This proof supposes that $D \neq 0$. But if we suppose the elements of A to vary so that D tends to zero, we see in the limit that $A_2 = 0$ when $D = 0$, provided that $n > 2$. If $n = 2$, $A_2 = A$ for all values of D.

Again, we can clearly write $A_{2r} = D^R A$, where R is a function of r. Now,

$$\text{Det } A_{2r} = D^{(n-1)2^r}, \text{ and Det } (D^R A) = D^{nR+1},$$

whence $$R = \{(n-1)2^r - 1\}/n.$$

So also $$A_{2r+1} = (\text{Det } A_1)^R A_1 = D^{(n-1)R} A_1.$$

M. F. EGAN.

2588. *Concerning arrangements in a circle.*

It is well-known, and simple to establish, that the number of arrangements *in a line* of n objects, of which p are alike of one kind, q alike of another kind, and so on, is

$$\frac{n!}{p!\,q!\,\ldots}.$$

The derivation of a corresponding formula for the number of arrangements *in a circle* presents rather more difficulty, and it is the purpose of this note to derive such a formula.

Let us denote by d the highest common factor of $p, q, \ldots$; and let the factors of d in descending order of magnitude be

$$d = d_0, d_1, d_2, \ldots, d_m = 1.$$

Write $p = p_0 d_0 = p_1 d_1 = p_2 d_2 = \ldots = p_m d_m$, $(p_m = p)$, and similarly etc.

We shall call an arrangement (linear or circular) d_i-symmetrical if it consists of a chain of exactly d_i identical arrangements of $p_i + q_i + \ldots$ objects. (*Example.* *aabaab* is a 2-symmetrical arrangement of the four letters *a* and the two letters *b*.) Clearly every arrangement is d_i-symmetrical for some $i = 0, 1, \ldots, m$.

Denote by K_i the number of linear d_i-symmetrical arrangements ; it is easily seen that

$$K_i = \frac{(n/d_i)!}{p_i!\,q_i!\,\ldots} - \sum_{j=0}^{i-1} S_{ij} K_j,$$

$i = 0, 1, \ldots, m$, where $S_{ij} = 1$ or 0 according as d_i does or does not divide d_j. These equations may be written in the form

$$\begin{aligned}
K_0 &= l_0 \\
S_{10} K_0 + K_1 &= l_1 \\
S_{20} K_0 + S_{21} K_1 + K_2 &= l_2 \\
&\ldots\ldots\ldots\ldots \\
S_{m0} K_0 + S_{m1} K_1 + S_{m2} K_2 + \ldots + K_m &= l_m,
\end{aligned}$$

with $l_i = \left(\frac{n}{d_i}\right)! \Big/ p_i!\,q_i!\,\ldots$; and their solution is

$$K_i = \begin{vmatrix} 1 & 0 & \ldots & l_0 & \ldots & 0 \\ S_{10} & 1 & \ldots & l_1 & \ldots & 0 \\ S_{20} & S_{21} & \ldots & l_2 & \ldots & 0 \\ \ldots\ldots & \ldots\ldots & & \ldots\ldots & & \\ S_{m0} & S_{m1} & \ldots & \underset{(i+1)}{l_m} & \ldots & 1 \end{vmatrix}, \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$i = 0, 1, \ldots, m$.

Now a d_i-symmetrical circular arrangement gives rise to exactly $\binom{n}{d_i}$ different d_i-symmetrical linear arrangements. Therefore the total number of circular arrangements of the n objects is equal to

$$\sum_{i=0}^{m} \binom{d_i}{n} K_i,$$

where the K_i are given by (1). This is the desired result.

HAZEL PERFECT.

2589. *A geometrical interpretation of the formula for the derivative of a function of a function.*

Let x, y, z be coordinates in a trirectangular frame F and let dx, dy, dz be coordinates in a parallel frame F_P with origin at the point P with coordinates $x, g(x), f(g(x))$. P is a point on the curve C with equations $y=g(x)$, $z=f(y)$. The equations of the tangent to C at P, relative to the frame F_P, are

$$\frac{dx}{1}=\frac{dy}{Dg(x)}=\frac{dx}{Df(g(x))}$$

and the equations of the projection of this tangent on the plane $dx=0$ are

$$\frac{dy}{Dg(x)}=\frac{dz}{Df(g(x))}, \quad dx=0;$$

but the equations of the tangent at P to the projection of C on $dx=0$ are

$$dz=f'(y)\,dy, \quad dx=0,$$

relative to F_P, and so the formula

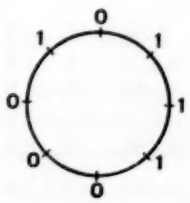
$$Df(g(x))=f'(g(x))Dg(x)$$

for the derivative of the composite function $f(g(x))$ is a consequence of the fact that tangency is preserved under orthogonal projection. Of course it is only the *formula* for the derivative of $f(g(x))$ which can be obtained in this way; we have already assumed the *differentiability* of $f(g(x))$ in speaking of the tangent to the curve C.

R. L. GOODSTEIN.

2590. *A permutation problem.*

In the design of servo-mechanisms the following problem arises: Can we label each of 2^n points round a circle with a 0 or a 1 in such a way that each set of n consecutive points round the circle has a unique number (reading anti-clockwise)? For instance with 8 points the distribution



gives the eight possible sets of three consecutive points the numbers 000, 001, 011, 111, 110, 101, 010, 100. For sixteen points the arrangement

0111101100101000

provides a solution.

If $a_0a_1a_2\ldots$ is any succession of 0's and 1's the numbers

$$A_r = a_r a_{r+1}\ldots a_{r+n-1}$$

$$A_{r+1} = a_{r+1}a_{r+2}\ldots a_{r+n}$$

of successive sets of n consecutive digits, satisfy the congruence

$$A_{r+1} = 2A_r + a_{r+n} \pmod{2^n}.$$

The problem may therefore be expressed in the following form : Can we find a permutation $A_0, A_1, \ldots, A_{2^n-1}$ of the numbers $0, 1, \ldots, 2^n - 1$ such that, for $0 \leqslant r \leqslant 2^n$, $0 \leqslant A_{r+1} - 2A_r \leqslant 1 \pmod{2^n}$, $A_{2^n} = A_0$?

We shall show that the desired permutation may be defined as follows.

$$A_0 = 0,\ A_{r+1} = 2A_r + 1 \pmod{2^n}$$

if there is no $s \leqslant r$ for which $A_s = 2A_r + 1 \pmod{2^n}$; otherwise $A_{r+1} = 2A_r \pmod{2^n}$ unless there is an $s \leqslant r$ for which $A_s = 2A_r \pmod{2^n}$, in which case A_r is the last term.

In other words the rule for generating the permutation is, starting with 0, *double and add one* (and reduce modulo 2^n) unless this yields a number which has already occurred, in which case we simply double (and reduce), the series stopping when neither doubling nor doubling and adding one produces a number which has not already occurred.

The sequence of A's cannot contain more than 2^n terms, since each A_r is a number from 0 to $2^n - 1$ and no two A_r, A_s are equal. We have therefore to prove that the sequence does in fact contain 2^n terms, and that the last term is 2^{n-1} (to join the cycle on to the initial 0).

Theorem 1. *The sequence of A's ends with the number* 2^{n-1}.

For if l is the last term, then both $2l$ and $2l+1$ (reduced mod 2^n) precede l. (or the sequence would not stop at l). But $2l, 2l+1$ can only follow l or $l + 2^{n-1}$, and since l is last and $l + 2^{n-1}$ does not occur twice, one of $2l, 2l+1$ has no predecessor, and is therefore the first term ; thus $2l = 0 \pmod{2^n}$ and therefore, since $l \neq 0$, $l = 2^{n-1}$.

Theorem 2. *If the number r does not occur in the sequence of A's then* $2r$ *is also missing.*

For $2r+1$ precedes $2r$, and since r is absent $2r+1$ follows $r + 2^{n-1}$, and there is no predecessor for $2r$.

Theorem 3. *All the numbers* $0, 2, 4, \ldots, 2^{n-1}$ occur in the sequence of A's.

For if 2^k were absent then $2^{k+1}, 2^{k+2}, \ldots, 2^{n-1}$ would all be absent, whereas we know that 2^{n-1} occurs.

Theorem 4. *All the numbers from* 0 *to* $2^n - 1$ *occur in the sequence of A's.*

For if $m = 2^r(2k+1)$ were absent (where r may be zero) then $2^s m$ would be absent for any s, and so the number congruent with $2^{n-1}(2k+1)$ would be absent, i.e. 2^{n-1} would be absent, contradicting Theorem 1.

Solutions of this problem using topological methods and the properties of finite graphs have previously been given by de Bruijn, I. J. Good and D. Rees ; de Bruijn proved that there are $2^{f(n)}, f(n) = 2^{n-1} - n$, solutions.

R. L. GOODSTEIN.

2591. *The expansions of the elementary functions.*

The Mathematical Association report on the teaching of Calculus in schools, in its chapter on expansions, illustrates a number of ways in which the expansions of the circular and logarithmic functions may be obtained in a first course in Calculus without the use of Taylor's theorem. Assuming a definition of the circular functions which yields the fundamental properties

$$D \sin x = \cos x, \quad D \cos x = -\sin x$$

(e.g. $\arctan x = \int_0^x \{1/(1+t^2)\}\,dt$, $\sin x = 2\tan(x/2)/\{1+\tan^2(x/2)\}$,

$$\cos x = \{1-\tan^2(x/2)\}/\{1+\tan^2(x/2)\},$$

completing the definitions by taking $\tan x$ to be periodic with period π, and $\sin \frac{\pi}{2} = 1$, $\sin\left(-\frac{\pi}{2}\right) = -1$, $\cos\left(\pm\frac{\pi}{2}\right) = 0$), and assuming that the integral of a positive function is positive, the report shows, with full rigour, that $\sin x$ lies between any consecutive two of the polynomials

$$x,\ x - \frac{x^3}{3!},\ x - \frac{x^3}{3!} + \frac{x^5}{5!},\ x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

and $\cos x$ between any consecutive two of

$$1,\ 1 - \frac{x^2}{2!},\ 1 - \frac{x^2}{2!} + \frac{x^4}{4!},\ 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots.$$

It remains only to prove that $x^n/(n!) \to 0$ to complete the identification of $\sin x$ with the series

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

and $\cos x$ with the series

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

and this readily follows from the simple inequality

$$\frac{|x|^{n+1}}{(n+1)!} < \tfrac{1}{2}\,\frac{|x|^n}{n!}$$

which holds for $n > 2|x| - 1$.

What is perhaps even more important the proof supplies a simple bound for the error in taking the polynomial

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^{n-1}\,\frac{x^{2n-1}}{(2n-1)!}$$

for $\sin x$, by showing that the error is less than $|x|^{2n+1}/(2n+1)!$, i.e. less than the absolute value of the first term neglected in the series, with a similar estimate of the error in taking

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + (-1)^n\,\frac{x^{2n}}{(2n)!}$$

for $\cos x$.

A similar, but less complete, discussion is given for the logarithmic and exponential series, but the report does not deal with the binomial series, and the main purpose of this note is to describe an elementary treatment of the

binomial series which has recently occurred to me. I shall however start with a few remarks about the logarithmic and exponential series.

Since $$1+t+t^2+\ldots+t^{n-1}=\frac{1}{1-t}-\frac{t^n}{1-t}$$

the error in taking

$$x+\frac{x^2}{2}+\ldots+\frac{x^n}{n}$$

for $-\log(1-x)$ is $$\int_0^x \{t^n/(1-t)\}\,dt,$$

but, if $0\leqslant t<1$, $nt^n<1+t+t^2+\ldots+t^{n-1}<1/(1-t)$ so that

$$0<\int_0^x \frac{t^n}{1-t}\,dt<\frac{1}{n}\int_0^x \frac{1}{(1-t)^2}\,dt=\frac{1}{n}\cdot\frac{x}{1-x},$$

which shows that the error in taking

$$x+\frac{x^2}{2}+\ldots+\frac{x^n}{n}$$

for $\log(1-x)$ is less than $x/n(1-x)$, if $0\leqslant x<1$.

Similarly, from the identity

$$1-t+t^2-\ldots+(-t)^{n-1}=\frac{1}{1+t}-\frac{(-t)^n}{1+t}$$

we find that the error in taking

$$x-\frac{x^2}{2}+\ldots-\frac{(-x)^n}{n}$$

for $\log(1+x)$ is

$$\int_0^x \frac{t^n}{1+t}\,dt$$

and for $0\leqslant x<1$ this error is less than

$$\int_0^x \frac{t^n}{1+t}\,dt<\frac{1}{n+1}.$$

These bounds decide the convergence of both the series

$$\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots$$

$$-\log(1-x)=x+\frac{x^2}{2}+\frac{x^3}{3}+\ldots$$

for the range $-1<x<1$, but the fact that

$$\log 2=1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\ldots$$

cannot be obtained in this way. However

$$\sum_{r=0}^{n-1}\frac{1}{1+r/n}\cdot\frac{1}{n}\to\int_1^2\frac{1}{x}\,dx=\log 2$$

i.e. $$\frac{1}{n}+\frac{1}{n+1}+\ldots+\frac{2}{2n-1}\to\log 2,$$

and so

$$1-\frac{1}{2}+\frac{1}{3}-\ldots+\frac{1}{2n-1}=1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{2n-1}-2\left[\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\ldots+\frac{1}{2n-2}\right].$$

$$=\frac{1}{n}+\frac{1}{n+1}+\ldots+\frac{1}{2n-1}\to\log 2.$$

For the exponential series we observe that

$$D\left(1+x+\frac{x^2}{2!}+\ldots+\frac{x^n}{n!}\right)e^{-x}=-\frac{x^n}{n!}e^{-x}$$

showing that

$$\left(1+x+\frac{x^2}{2!}+\ldots+\frac{x^n}{n!}\right)e^{-x}$$

decreases for $x>0$, and therefore for $x>0$,

$$1+x+\frac{x^2}{2!}+\ldots+\frac{x^n}{n!}<e^x,$$

and similarly

$$D\left(1+x+\frac{x^2}{2!}+\ldots+\frac{x^{n-1}}{(n-1)!}+2\frac{x^n}{n!}\right)e^{-x}=\frac{x^{n-1}}{(n-1)!}e^{-x}\left(1-\frac{2x}{n}\right)>0,$$

for $n>2x$, and therefore, for $n>2x>0$

$$1+x+\frac{x^2}{2!}+\ldots+\frac{x^n}{n!}<e^x<\left(1+x+\frac{x^2}{2!}+\ldots+\frac{x^n}{n!}\right)+\frac{x^n}{n!}$$

which shows that (for $x>0$) the error in taking

$$1+x+\frac{x^2}{2!}+\ldots+\frac{x^n}{n!}$$

for e^x is less than $x^n/n!$ (for $n>2x$).

Moreover, for $x>0$,

$$D\left(1-x+\frac{x^2}{2!}-\ldots+\frac{(-x)^n}{n!}\right)e^x=\frac{(-x)^n}{n!}e^x$$

which is positive or negative according as n is odd or even, and so comparing the value for $x>0$ with the value at $x=0$ we see that e^{-x} lies between

$$1-x+\frac{x^2}{2!}-\ldots+\frac{(-x)^n}{n!}\quad\text{and}\quad 1-x+\frac{x^2}{2!}-\ldots+\frac{(-x)^{n+1}}{(n+1)!}$$

(for any n), which shows that the error in taking

$$1+x+\frac{x^2}{2!}+\ldots+\frac{x^n}{n!}$$

for e^x is less than $|x|^{n+1}/(n+1)!$ when $x<0$.

The foregoing method is also successful in dealing with the binomial series. Denoting as usual the binomial coefficient

$$m(m-1)(m-2)\ldots(m-r+1)/r!$$

by $\binom{m}{r}$, we may readily verify that the derivative of

$$(1+x)^{-m}\left\{1+\binom{m}{1}x+\binom{m}{2}x^2+\ldots+\binom{m}{r}x^r\right\}$$

is $$(r-m)\binom{m}{r}\frac{x^r}{(1+x)^{m+1}}$$

and the derivative of

$$(1+x)^{-m}\left\{1+\binom{m}{1}x+\binom{m}{2}x^2+\ldots+\binom{m}{r}x^r+\binom{m}{r+1}(r+1)\,x^{r+1}\right\}$$

is $$(m-r)\binom{m}{r}\frac{x^r}{(1+x)^{m+1}}\{r(1+x)-(m-1)\,x\}\,;$$

thus if m is not a positive integer and if $n>-1\,(x\neq 0)$, and $r>(m-1)x/(1+x)$ these two derivatives have opposite signs and therefore $(1+x)^m$ lies between

$$1+\binom{m}{1}x+\binom{m}{2}x^2+\ldots+\binom{m}{r}x^r$$

and $$1+\binom{m}{1}x+\binom{m}{2}x^2+\ldots+\binom{m}{r}x^r+\binom{m}{r+1}(r+1)\,x^{r+1}$$

so that the error in taking

$$1+\binom{m}{1}x+\binom{m}{2}x^2+\ldots+\binom{m}{r}x^r \quad \text{for} \quad (1+x)^m$$

is less than $$\binom{m}{r+1}(r+1)\,|x|^{r+1} \quad (\text{for } x>-1).$$

It remains only to show that, for $-1<x<1$, this bound tends to zero as r tends to infinity. The bound may be written mxu_r where

$$u_r=\binom{m-1}{r}x^r$$

so that

$$\frac{u_{r+1}}{u_r}=-\left(1-\frac{m}{r+1}\right)x\,;$$

it follows that, for $r\geqslant m$,

$$\left|\frac{u_{r+1}}{u_r}\right|=\left\{1-\frac{m}{r+1}\right\}|x|\,.$$

Write $\delta=(1-|x|)/(1+|x|)$ so that $0<\delta<1$ if $|x|<1$, then

$$|x|=(1-\delta)/(1+\delta)$$

and, for $r+1>-m/\delta$, $1-m/(r+1)<1+\delta$, whence

$$0<\left\{1-\frac{m}{r+1}\right\}|x|<1-\delta$$

so that $$\left|\frac{u_{r+p}}{u_r}\right|<(1-\delta)^p$$

which proves that $u_n\to 0$ as $n\to\infty$.

R. L. GOODSTEIN.

2592. *Knight's rotationally symmetric re-entrant tour.*

There already exist many unsolved problems connected with knights' tours on a board of n^2 cells (see, for instance, Rouse Ball's *Mathematical Recreations and Essays*, 11th edition, pp. 174–185); to these I should like to add another which may be of interest, though unfortunately only relevant when n is singly-even.

It is first required to find a re-entrant knight's tour such that any quarter of the path (i.e. any set of $\frac{1}{4}n^2$ successive cells of the path) is obtainable by a rotation through one right-angle of the previous quarter; more precisely, denoting the square in the x-th file and y-th rank by (x, y), then if (x, y) is the r-th cell of the path we require that $(y, n-x+1)$ is the $(\frac{1}{4}n^2+r)$-th cell of the path.

It is clear that n must be even, and so that $x+y$, $y+n-x+1$ are of opposite parity; this implies that an odd number of moves is required to get from (x, y) to $(y, n-x+1)$, and so that $\frac{1}{4}n^2$ is odd; hence n must be singly even.

In practice it is possible to trace such a tour with little difficulty, by adding to each of four separate (rotationally symmetric) paths simultaneously and using Warnsdorff's rule that the knight must always be moved to one of the cells from which it will command the fewest squares not already traversed.

The main problem is to determine how many fundamentally different such tours there are for a given (singly-even) n; for $n=6$ the answer of four can be easily obtained by an exhaustive process, but for $n \geqslant 10$ such a method would be also exhausting. The four fundamentally different solutions in the $n=6$ case are indicated below (only one quarter of each path being shown, the remainder being given by the above rule).

. . 4 . . .	. . 4 . . .	. . 4 . . .	. . . . 4 .
3 . . . 5 .	3	3 . . . 5 .	. . . . . .
. 9	. 9 . 5 . .	. 9	. 9 . 3 . 5
. 2 . 8 . 6	. 2	. 2 . 6 . .	. 2 . . . .
. . . . . .	8 . . . 6 .	8	8 . . . 6 .
1 . . . 7 .	1 . 7 . . .	1 . 7 . . .	1 . 7 . . .

H. APSIMON.

2593. *A further note on the nedians of a plane triangle.*

In note 2392, although it is not expressly stated that n is integral, this would seem to be implied; and it is perhaps worth while to note that the removal of this unnecessary restriction allows certain generalisations. Replacing BC/n by $k.BC$ does not affect the mathematics; and k may assume any value, including negative values and values greater than unity. The area of the nedian triangle is now given by

$$\text{area } A'B'C' : \text{area } ABC = (2k-1)^2 : k^2-k+1,$$

and it is interesting to observe that as k approaches infinity the area of the nedian triangle approaches four times that of the original triangle—a result which becomes obvious when the limiting diagram is constructed. Negative values of k may be replaced by the corresponding positive values for the "backward" nedians (i.e. $|k|+1$), so give no trouble in analytical investigation.

The area of the "nedian hexagon" is of some interest. It is

$$2(1-2k)^2/(2-k)(1+k).$$

It obviously vanishes when $k=\frac{1}{2}$, and rather more surprisingly, becomes infinite when $k=2$ or -1. Once again a diagram quickly clears up the mystery.

If different nedians are taken from each vertex, k being replaced by x, y and z for AF', BD' and CE' respectively, we are dealing with any three lines, one through each vertex, and the vanishing of the nedian triangle, now of area $ABC \times (1 - \Sigma y(1-z)/(yz - z + 1))$ may be used to establish all the usual concurrences. For example, $x = 1 - z$, $y = \frac{1}{2}$, and similar substitutions, verify the concurrences in figure 3 of the note; $x = b \cos A/c$, etc., will give the altitudes; and so on. (Needless to say, this approach is not recommended in preference to the usual proofs!)

A slight improvement in the various formulae seems to be provided by letting $k = AF''/F''B$ instead of AF''/AB. But the slight gain did not appear to justify altering the shape of the results given in the original article.

Another useful addition to the notation is the use of x' for $1 - x$, etc. With this, the area of the nedian hexagon in the general case becomes

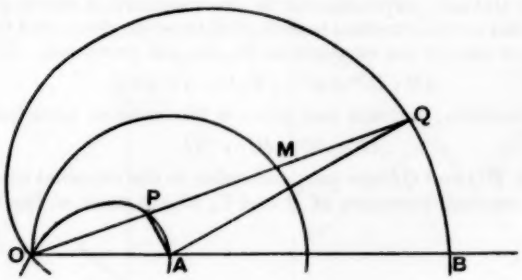
$$\triangle ABC(1 - \Sigma x^2y'/(1 - x'y') - \Sigma xy'^2/(1 - xy)).$$

H. R. CHILLINGWORTH.

2594. *Variation on Archimedes.*

The straight line $\theta = 2\alpha$ meets the limaçon $r = 1 + 2\cos\theta$ again at P, Q $(OP < OQ)$; A, B are the points $(-1, \pi)$ and $(3, 0)$ respectively. Then

$$\angle PAQ = \tfrac{1}{2}\pi, \quad \angle OQA = \tfrac{1}{3}\angle QAB.$$



Let the circle $r = 2\cos\theta$ meet OQ again at M. Then

$$1 = MQ = MP = AM.$$

Thus $\angle PAQ = \tfrac{1}{2}\pi$.

Further, $2\alpha = \angle AOM = \angle OMA = 2\angle OQA$.

Hence $\angle QAB = 3\alpha$,

and so $\angle OQA = \tfrac{1}{3}\angle QAB$.

F. BRIERLEY.

2595. *The angle-bisector ratio theorems: proofs based on the concurrencies.*

AD bisects the angle A and meets BC in D: to prove that $AB{:}AC = DB{:}DC$.

In the hope of using Thales' theorem concerning "sides divided proportionally", cut off, along BA and CA as in Fig. 1 or along AB and AC produced, as in Fig. 2, segments BP and CQ equal respectively to BD and CD.

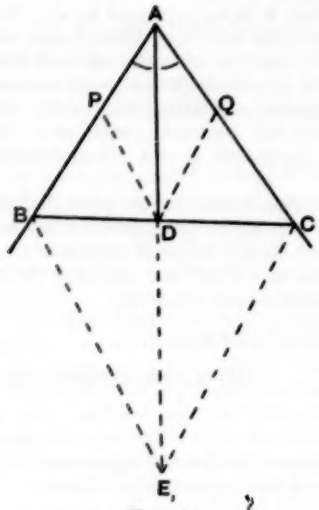


FIG. 1.

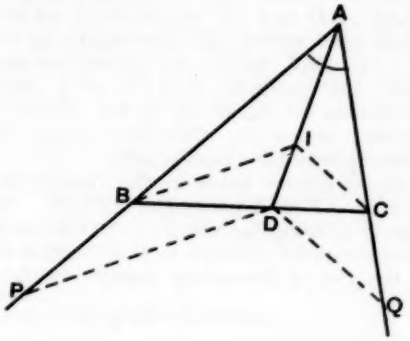


FIG. 2.

In order to make use of the isosceles triangles, join DP and DQ. Then, in Fig. 1, PD and QD are perpendicular to the bisectors of the angles B and C, and so are parallel to the external bisectors of these angles ; and these external bisectors meet at one of the ex-centres, E_1, on AD produced. Then

$$AB : BP = AE_1 : E_1D = AC : CQ,$$

using Thales' theorem, perhaps not quite in the manner anticipated. Hence

$$AB : AC = BD : CD.$$

And in Fig. 2, PD and QD are perpendicular to the external bisectors and so parallel to the internal bisectors of B and C, which meet at the incentre I on AD ; etc.

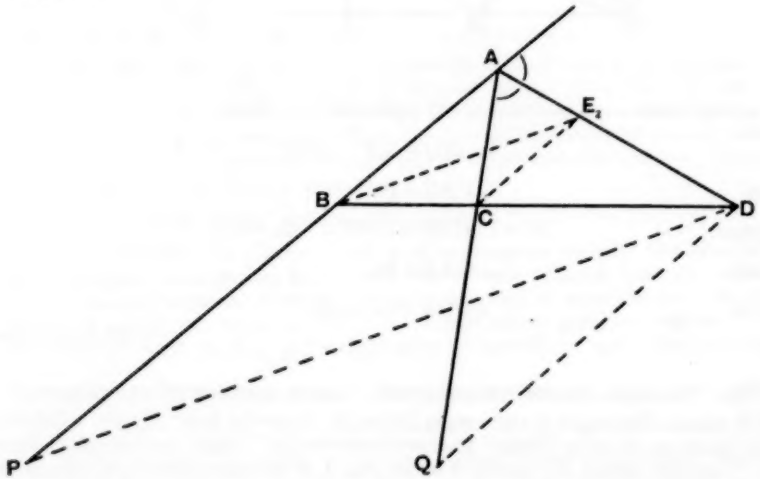


FIG. 3.

For the case in which AD bisects the exterior angle at A, one of the possible configurations is shown in Fig. 3, where BP and CQ, equal respectively to BD and CD, are cut off along AB and AC produced. (The alternative, corresponding to Fig. 1, is left to the reader.) We find that PD and QD are parallel respectively to the internal bisector of B and the external bisector of C, which meet at the ex-centre E_1 on AD.

Independent proofs of the converses are on similar lines. For example, if in Fig. 1 we are given $AB : AC = DB : DC$, the construction as before brings in PD and QD which are parallel to the external bisectors of the angles B and C. If these bisectors meet AD in F and G respectively, then we find that F and G divide AD in the same ratio ($AB : BD$ or $AC : CD$); that is, F and G coincide at a point on AD which, being common to the two external bisectors, must lie on the internal bisector of A. So AD bisects the angle A.

A. G. SILLITTO.

2596. *Pythagoras' theorem and its converse.*

Mr. J. E. Blamey in Note 2337 and Prof. V. Thébault in Note 2372 have given proofs of the converse which are independent of the theorem. An equally worth-while problem, for the teacher of young children, is to devise proofs which will evoke response from the spatial intuition of the learner. In the case of the Theorem, Euclid's proof, and some of its variants, do tend to

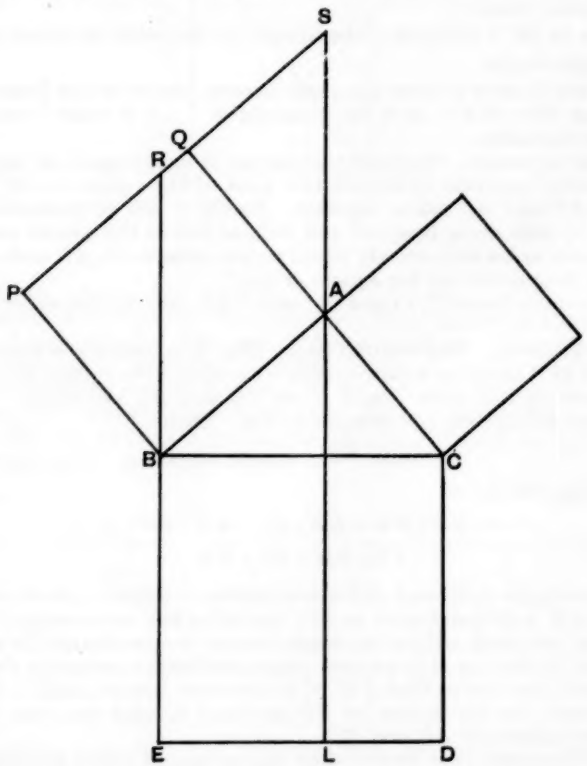


FIG. 1.

build up a conviction of the equality of the *areas* under discussion; while in the proof by ratio attributed to Pythagoras, and the various proofs of the Converse, the reference to area is wholly indirect and implicit.

In each of the two arguments outlined below, the proofs of Theorem and Converse are based on direct perception of squares and their areas and the equivalence of areas; and any instructional value they have resides therein.

I. In Fig. 1, ABC is any triangle, with squares described on the sides: EB produced cuts PQ in R; LAS is parallel to EBR.

If we compare the triangles BAC and BPR, we find that according as

$$\angle BAC \gtreqless \angle BPR$$

then $$BC \gtreqless BR,$$

that is $$BE \gtreqless BR,$$

and hence rectangle $BL \gtreqless$ parallelogram $ABRS$

(these being between the same parallels),

and so rectangle $BL \gtreqless$ square on AB.

Similarly rectangle $CL \gtreqless$ square on AC,

also according as $\angle BAC$ is obtuse, right or acute.

By addition, we assert the theorem as a border-line case of equality between two inequalities, thus:

the square on $BC \gtreqless$ the sum of the squares on the other two sides, according as $\angle A \gtreqless$ a right angle.

The converse follows at once from the theorem stated in this form: for if it be given that $BC^2 = BA^2 + AC^2$, the possibilities "$\angle A$ is acute" and "$\angle A$ is obtuse" are excluded.

II. In this approach, the young geometer is encouraged to argue thus: "There is some question of the relative sizes of the square on BC, and the squares on AB and AC taken together. Surely it will be profitable to construct, side by side along base BC and each as tall as the square on BC, two rectangles with areas respectively equal to the squares on AB and AC; and *see* whether they cover the big square or not."

We shall require Euclid I, 43 and 44; and VI, 2, and the properties of similar triangles.

First the Theorem. The original figure (Fig. 2) contains the triangle ABC right-angled at A, and the square on BC. Construct the square $EFGH$, equal to the square on AB, with F and H on DE and BE produced. Complete $BEFK$, draw KEL, then LM parallel to EB. Then

$$\text{rectangle } EM = \text{square on } AB.$$

Also, denoting BM by x,

$$c : x = KB : BM = KE : EL = KF : FG = a : c,$$

that is, $$AB : BM = BC : AB.$$

Hence the triangles ABC and MBA are similar, $\angle BMA = \angle BAC =$ one right angle, and AM is perpendicular to BC, that is, AMN is a straight line.

If we now construct within the angle formed by producing CD and ED a square equal to that on AC, we can obtain similarly a rectangle $DCM'N'$ of the same area, and prove that $AM'N'$ is the same line as AMN. The result follows at once, for the square on BC has been divided into two rectangles equal to the squares on AB and AC.

Now the Converse (but notice that the argument which follows does not depend on the foregoing, and could just as well precede it).

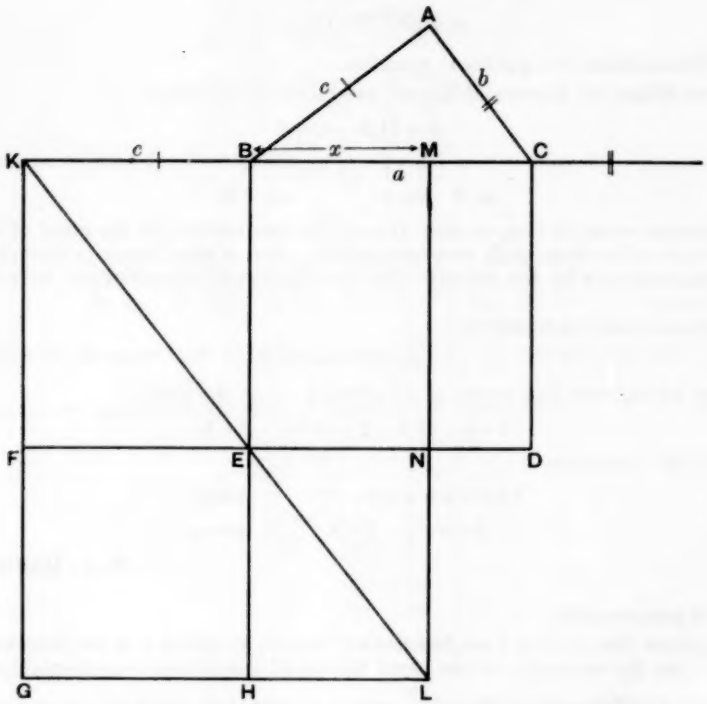


FIG. 2.

It is given that $BC^2 = AB^2 + AC^2$.

The rectangle EM, equal to the square on AB, is drawn as above, and the similarity of the triangles ABC and MEA established. Then the rectangle with side CD, equal to the square on AC, is drawn, and as, by hypothesis, this rectangle occupies the rest of the square on BC, it is also bounded by MN, and the triangles ABC and MAC are found to be similar. Hence the angles BAC, BMA and CMA are all equal ; and angles BMA and CMA are supplementary Hence the angle BAC is a right angle.

A. G. SILLITTO.

1853. The Fairey Delta 2 supersonic research aircraft . . . may prove to be the fastest jet-propelled aircraft that has yet flown, and will show that no disturbance is felt when passing through the sound barrier in level flight, provided this is done at sufficient speed.—*Daily Telegraph*, October 8th, 1954. [Per Mr. A. J. Cole.]

1854. Women know that day-dreaming gets you nowhere. That's why Newton and not Mrs. Newton discovered gravity.—*Sunday Express* [Per Mr. H. G. Woyda.]

PSEUDARIA

8. *The solution of a quadratic equation.*

A candidate (at Advanced Level!) produced the solution

$$(x+3)(2-x)=4$$

$$\therefore\ x+3=4 \qquad \therefore\ x=1,$$

$$\text{or } 2-x=4 \qquad \therefore\ x=-2.$$

His answers were, in fact, correct, though the examiner took the point of view that luck rather than skill was responsible. But a new terror is introduced into mathematics by the remark that *every* quadratic equation can be solved in this way :

Suppose that the equation is

$$x^2-(\alpha+\beta)x+\alpha\beta=0.$$

It may be expressed in terms of $1+\alpha$ and $1-\beta$ in the form

$$(1+\alpha-x)(1-\beta+x)=\alpha-\beta+1,$$

giving the " solution "

$$1+\alpha-x=\alpha-\beta+1 \qquad \therefore\ x=\beta,$$

$$\text{or} \qquad 1-\beta+x=\alpha-\beta+1 \qquad \therefore\ x=\alpha.$$

E. A. MAXWELL.

9. *A non-function.*

To prove that, *if x, y, z are functions of u, v, w, so that u, v, w are functions of x, y, z* (as, for example, in the usual theory of curvilinear coordinates), *then the partial differential coefficient $\dfrac{\partial x}{\partial u}$, when expressed in terms of x, y, z, is independent of x.*

For

$$\begin{aligned}\frac{\partial}{\partial x}\left(\frac{\partial x}{\partial u}\right)&=\frac{\partial^2 x}{\partial x\,\partial u}\\ &=\frac{\partial^2 x}{\partial u\,\partial x}\\ &=\frac{\partial}{\partial u}\left(\frac{\partial x}{\partial x}\right)\\ &=\frac{\partial}{\partial u}(1)\\ &=0.\end{aligned}$$

Hence $\dfrac{\partial x}{\partial u}$ is independent of x.

E. A. MAXWELL.

10. *There are no mutually orthogonal surfaces.*

Suppose that, indeed, there existed surfaces

$$u(x, y, z)=a,\ v(x, y, z)=b,\ w(x, y, z)=c$$

each cutting each of the others orthogonally. Then, since u, v and u, w are orthogonal pairs,

$$\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}+\frac{\partial u}{\partial z}\frac{\partial v}{\partial z}=0,$$

$$\frac{\partial u}{\partial x}\frac{\partial w}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial w}{\partial y}+\frac{\partial u}{\partial z}\frac{\partial w}{\partial z}=0,$$

so that, on solving for the ratios $\frac{\partial u}{\partial x} : \frac{\partial u}{\partial y} : \frac{\partial u}{\partial z}$,

$$\frac{\frac{\partial u}{\partial x}}{\frac{\partial v}{\partial y}\frac{\partial w}{\partial z}-\frac{\partial v}{\partial z}\frac{\partial w}{\partial y}}=\frac{\frac{\partial u}{\partial y}}{\frac{\partial v}{\partial z}\frac{\partial w}{\partial x}-\frac{\partial v}{\partial x}\frac{\partial w}{\partial z}}=\frac{\frac{\partial u}{\partial z}}{\frac{\partial v}{\partial x}\frac{\partial w}{\partial y}-\frac{\partial v}{\partial y}\frac{\partial w}{\partial x}}.$$

Multiply numerators and denominators by $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$ respectively, and add; we obtain an expression for each of these ratios in the form

$$\frac{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2}{\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}}$$

In the notation of Jacobians, this gives three relations, of which a typical one is

$$\frac{\partial u}{\partial x}\frac{\partial(u,v,w)}{\partial(x,y,z)}=\left\{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2\right\}\frac{\partial(v,w)}{\partial(y,z)}.$$

Moreover, the partial differential coefficients are subject to the usual relations such as

$$\frac{\partial u}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial u}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial u}=1,$$

$$\frac{\partial u}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial v}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial v}=0,$$

$$\frac{\partial u}{\partial x}\frac{\partial x}{\partial w}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial w}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial w}=0.$$

Solve for $\frac{\partial u}{\partial x}$ by multiplying by $\frac{\partial(y,z)}{\partial(v,w)}, \frac{\partial(y,z)}{\partial(w,u)}, \frac{\partial(y,z)}{\partial(u,v)}$ and adding. Thus

$$\frac{\partial u}{\partial x}\frac{\partial(x,y,z)}{\partial(u,v,w)}=\frac{\partial(y,z)}{\partial(v,w)}.$$

Now multiply left-hand sides and right-hand sides of the two relations for $\frac{\partial u}{\partial x}$, remembering the identities

$$\frac{\partial(u,v,w)}{\partial(x,y,z)}\frac{\partial(x,y,z)}{\partial(u,v,w)}=1, \quad \frac{\partial(v,w)}{\partial(y,z)}\frac{\partial(y,z)}{\partial(v,w)}=1.$$

Then $$\left(\frac{\partial u}{\partial x}\right)^2=\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2,$$

so that $$\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2=0.$$

Similarly $$\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial u}{\partial x}\right)^2=0,$$

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2=0.$$

Hence $$\frac{\partial u}{\partial x}=0,\quad \frac{\partial u}{\partial y}=0,\quad \frac{\partial u}{\partial z}=0,$$

so that *the function $u(x, y, z)$ does not depend on any of the variables x, y, z.*

Identical argument produces the corresponding result for $v(x, y, z)$ and $w(x, y, z)$, and so the three proposed surfaces do not actually exist.

E. A. MAXWELL.

11. *Odd logs.*

(*a*) $\log(-1)=\log(1/-1)=-\log(-1)$, hence $\log(-1)=0$.

(*b*) An arbitrary function can be resolved into the sum of two parts, one even and one odd, thus :

$$f(x)=\tfrac{1}{2}\{f(x)+f(-x)\}+\tfrac{1}{2}\{f(x)-f(-x)\}.$$

Apply this to the function $\log x$; we have

$$\log x=\tfrac{1}{2}\{\log x+\log(-x)\}+\tfrac{1}{2}\{\log x-\log(-x)\}$$

$$=\tfrac{1}{2}\log(-x^2)+\tfrac{1}{2}\log(-1).$$

Thus the odd part of the function $\log x$ is the constant $\frac{1}{2}\log(-1)$.

A. J. HOWIE.

Problem Bureau.

The problem Bureau welcomes requests for solutions of interesting problems of Cambridge Entrance Scholarship standard. The Bureau is under the direction of Dr. G. A. Garreau, 90 Wyatt Park Road, London S.W.2, and problems should be addressed to him.

1855. A metal . . . possesses unique . . . characteristics . . . . It is 90·98 per cent magnesium metal, 6 per cent aluminum, 3 per cent zinc, and 2 per cent manganese. One can see why even the sceptical . . . find this new metal more interesting. *Times Literary Supplement*, 19th September, 1954 (*American Writing Today*, p. lxvi). [Per Mr. Frank Sandon.]

1856. France-Vietnamese forces are now confined to a defence area at Dien Bien Phu only one-third the size of the perimeter they held at the outset of the fighting.—*The Observer*, 25th April, 1954. [Per Mr. G. N. Copley.]

REVIEWS

Table of the Gamma Function for Complex Arguments. Pp. xvi, 105. $2. 1954. Applied Mathematics Series, 34. (National Bureau of Standards, Washington)

The main table gives to 12 decimals, without differences, the real and imaginary parts of $\log_e \Gamma(x+iy)$ for $x=0(\cdot1)10$, $y=0(\cdot1)10$. It has one page for each of the 101 values of x. There is also a 2-page table of circular and hyperbolic sines and cosines of πx for $x=0(\cdot1)10$.

The main table is interpolable except in the neighbourhood of the origin. Interpolation in a bivariate table which gives so many figures in the function values at rather wide intervals of the arguments cannot be expected to be altogether easy; methods are provided in the Introduction, written by H. E. Salzer. The reader may be reminded that a table by J. P. Stanley and M. V. Wilkes, giving $1/\Gamma(x+iy)$ to fewer figures in a smaller domain but with a finer lattice, was reviewed in *Math. Gazette*, **38**, 56, 1954. In both tables the domain may be extended by use of the analytical properties of the gamma function. Various smaller tables with complex arguments are mentioned in the Bibliography.

The present table is neatly arranged and produced. The reviewer concurs heartily with the statement that "past experience indicates that a basic table to many places is very useful". An accurate fundamental table of this kind is indeed to be regarded as precious. The volume exemplifies the considerable amount of work now being done on tabulating for complex arguments various functions which have hitherto been tabulated chiefly for real arguments.

A. FLETCHER.

Science awakening. By B. L. VAN DER WAERDEN. Pp. 306. $5. 1954. (Noordhoff, Groningen)

Mathematik in deiner Welt. By K. MENNINGER. Pp. 233. DM 12.80. 1954. (Vandenhoeck & Ruprecht, Göttingen)

Mathematics claims a large part in the curriculum for the average pupil. We can hardly justify this on grounds of utility, since few of these pupils will ever use more than simple arithmetic; the old argument that "mathematics, more than any other school subject, trains the mind" has worn very thin and no mathematician would place much weight on it. But the position can and should be defended by the fact that mathematics is an integral and vital element of our civilisation; thus an education which should offer our children the keys to the culture of the modern world must, for that reason, give a prominent place to mathematics.

That the eminent algebraist, van der Waerden, is of this opinion is evident from the preface to his book, wherein he explains his reasons for writing an account of mathematics in Babylon, Egypt and Greece. Our life today, he says, is influenced for good and ill, physically, mentally, spiritually, by science and technology, and these depend on mathematics. Modern science is founded on Newton, "from a *historical* point of view . . . the most important figure of the 17th century" and the lines of thought which converge on Newton all come from Greece. Further, "the history of mathematics can not be detached from the general history of culture" (I should myself say that it can not be detached from general history); mathematics is "a domain of intellectual activity, intimately related not only to astronomy and mechanics, but also to architecture and technology, to philosophy and even to religion".

The author has taken pains to make his account as simple and as up-to-date as possible. He has made much use of Neugebauer's epoch-making work on

the Babylonians, and of other recent researches. But he is not a mere compiler, for he presents a number of new ideas, particularly concerning Thales and Theaetetus. He also has something new to say about the decline of Greek mathematics; but here he seems to desert his own principles in order to follow Zeuthen in tracing this decline to technical reasons, such as the inadequacy of geometric algebra. These are surely not inner causes but superficial symptoms: the real reason is to be found in the failure of the Greeks and the Romans to effect a real marriage of minds. The Hellene recoiled from the pushing, practical Latin, who in turn despised the avid intellectual inquisitiveness of the Hellene; the consequent divorce of theory from practice entailed the ultimate sterility of both.

While this volume does not replace Heath's *Manual*, it makes a most admirable supplement. The Noordhoff typography is well up to its own high standard, and there are 28 magnificently reproduced plates; these, as a rule, do not directly illustrate the text, a function performed by numerous line blocks, but they serve to remind us of the amazing versatility of the Greek genius, that "love of beauty without extravagance" to which Euclid and Archimedes contributed as much as Aeschylus and Pheidias.

Dr. Menninger's book is of a familiar type, but seems to me to be of an altogether higher class than many of its kind. There is no pretentious technological verbiage but in clear and simple language John Citizen is told how mathematics answers questions and solves puzzles, how it underlies his maps or his little flutter on the roulette wheel at Monte Carlo. There are numerous black and white illustrations, serious and not so serious, and a happy informality of approach which should recommend the book to many who would normally as lief read the telephone directory as a mathematical treatise. From the first chapters, "Zählen kann eine Kunst sein" and "Warum der Elefant kein Floh ist", the text proceeds cheerfully through football pools to a natural end in "Tod und Leben unter der mathematischen Brille". Altogether a pleasant and delightful little book. T. A. A. B.

The Japanese Abacus. Its use and theory. By TAKASHI KOJIMA. Pp. 102. $1. 1954. (Tuttle Co., Tokyo)

A few years ago a contest between the Japanese abacus and an electrically operated calculating machine saw Nippon first and electricity nowhere. This shock to the electronic age caused a revival of interest in the once widely used device, now so often in the West seen only as a nursery toy, and, in particular, in the Japanese abacus, used, so Mr. Kojima tells us, in 92% of all Japanese business calculations. Mr. Kojima's little book gives us the history of the Japanese instrument, and then expounds, in great detail, its use in carrying out the "four rules". He does not deal with the extraction of roots, since this process is not often required in business. The publisher adds a note to say that he can obtain Japanese abacuses for customers. Those of us who have deplored the inability of the average shop assistant to give correct change may wonder whether a wholesale importation is desirable.

T. A. A. B.

Researches and Studies. Pp. 99. 3s. 6d. 1955. (Institute of Education, University of Leeds)

Among the nine studies in this volume, two are of special interest to mathematical teachers, that by P. J. Wallis on the 17th century schoolmaster, Charles Hoole, and by Miss Wallbank on "The mathematical attainments of students entering training colleges". This latter is a topic of prime importance, and Miss Wallbank's essay was used as a basis for further discussion at a conference in Leeds on the teaching of mathematics. The essay and the

summary of the discussion may be warmly recommended to grammar school teachers. T. A. A. B.

Les fondements logiques des mathematiques. By E. W. Beth. 2nd edition. Pp. xv, 241. 2500 fr. 1955. (Gauthier-Villars, Paris)

The first edition of this highly informative book was reviewed in Vol. XXXV, pp. 143–4. The new edition has replaced typescript by print, and the value of the book has been further enhanced by the elimination of errors and misconceptions, by fuller references to recent work and above all by accounts of the new topological methods in the proofs of Gödel's completeness theorem and the Löwenheim-Skolem theorem, introduced by Rasiowa and Sikorski. R. L. Goodstein.

Électricité et Optique. By H. Poincaré. 2nd edition, rep. Pp. x, 641. 2000 fr. 1954. (Gauthier-Villars, Paris)

The conquest of the Continent, particularly of France, by Maxwell's electromagnetic theory was much expedited by the publication of Poincaré's Sorbonne lectures, of which we now have a reprint of the second edition. We need not attempt to describe those qualities of Poincaré's genius which served to complement the massive and sometimes rugged work of Maxwell, since Poincaré's own words in his preface have been quoted and commented upon by Professor Temple in his brilliant and witty inaugural lecture at Oxford, *The Classic and Romantic in Natural Philosophy*, with which every sensible reader of the *Gazette* must already be acquainted.

Forty years have passed since Poincaré's death, yet we still encounter the influence of the *Doctor Mirabilis* in many current mathematical developments. This classic is one of the buttresses of his fame and we must thank Mm. Gauthier-Villars for their pious restoration of it to available form.

T. A. A. B.

La sommation des séries divergentes. By M. Zamansky. Pp. 46. 700 fr. 1954. Mémorial des sciences mathématiques, 128. (Gauthier-Villars, Paris)

This is a summary of results on one aspect of the theory of divergent series. Some forty theorems, mostly well-known, are stated concerning the relative "strength" of various types of summability, with special reference to methods defined by a limit of the form

$$\lim_{x\to\infty} \Sigma g\left(\frac{k}{x}\right) a_k.$$

There are no proofs. The reviewer notes the tantalizing (unsubstantiated) statement that the theorem that *the Lambert method* (where $g(t) = te^{-t}/(1-e^{-t})$) *is included in* (i.e., weaker than) *the Abel method* (where $g(t) = e^{-t}$) "s'expose très simplement" by a technique of the author's. This theorem is known to include the prime number theorem, but the only proof of it so far published depends on an assumption not weaker than the prime number theorem itself.* L. S. B.

Le calcul symbolique à deux variables et ses applications. By L. Poli and P. Delerne. Pp. 77. 1000 fr. 1954. Mémorial des sciences mathématiques, 127. (Gauthier-Villars, Paris)

After reading this tract, the overall impression left is of so much condensed into such a small space. Perhaps it may be termed a small work of reference. This is borne out by the bibliography which contains a list of 84 books and papers, to 73 of which there are copious references in the text.

* See G. H. Hardy, *Divergent series*, (O.U.P., 1949), p. 373.

Chapter I extends the definition

$$p\int_0^\infty e^{-pt}F(t)\,dt$$

of the Laplace transform of $F(t)$ in one variable (called the image of $F(t)$ in CS_1) to

$$pq\int_0^\infty\int_0^\infty e^{-ps-qt}F(s,t)\,ds\,dt$$

(called the image of $F(s,t)$ in CS_2). Sections are devoted to existence theorems, differentiation and integration with respect to a parameter, and formulae of inversion, though in many cases the reader is referred to other works for a detailed treatment. Chapter II is devoted to operational formulae which are extensions in CS_2 of known formulae in CS_1. The next chapter concerns similar formulae which have no counterparts in CS_1. Applications of the previous results are given in the last chapter. They consist of (1) calculation of integrals, (2) partial differential equations, (3) Bessel functions of third order, (4) Laguerre polynomials in two variables. There is a short appendix on symbolic calculus in two variables.

A serious reader will need alongside him an adequate supply of writing materials, for this is not a book to be skimmed through lightly. However, if he wishes to know something of Laplace transforms in two variables, he will find the effort worthwhile. M. HUTTON.

Cours de Cinématique. Tome I. (Cinématique du point et du solide. Composition des mouvements.) 3rd edition. By RENÉ GARNIER. Pp. ix, 244. 4000 fr. 1954. (Gauthier-Villars, Paris)

The book reviewed is the third edition of the first volume of the author's three-volume treatise on kinematics. Garnier regards kinematics as a branch of rational mechanics, of interest for its own sake and for its applications to differential geometry. Throughout the whole treatise little attention is paid to applications to engineering.

The first volume contains a very lucid account of the elementary notions of kinematics, and is much easier to read than the succeeding volumes. For his tools the author uses a combination of vector analysis and the method of moving frames. The first chapter deals with the kinematics of a point, and includes many applications to the differential geometry of skew curves. The field of velocities, helicoidal motion, and the field of accelerations are studied in the second chapter. The third chapter deals with the composition of motions and includes several proofs of Coriolis' Theorem. This chapter also gives a particularly lucid account of the motion of a solid body when one of its points is fixed. The fourth and last chapter studies how the motion of a solid is determined when its field of velocities is known.

The book is well printed and illustrated—the general format is in keeping with the elegant exposition of the subject matter. T. J. WILLMORE.

Vorlesungen über Differential- und Integralrechnung, III. By A. OSTROWSKI. Pp. 475. Sw. fr. 78. 1954. (Birkhäuser, Basel)

This third volume concludes Professor Ostrowski's admirable treatise on the calculus. The first section adds some detail to the earlier account of integration of functions of a single variable ; among other matters, it tightens and extends the rather casual introductory discussion of partial fractions given in Vol. I.

The main topics, occupying some 300 pages, are the doctrines of multiple integrals and infinite integrals. At this point exposition of the calculus faces its most difficult task, the maintenance of a due balance between sound theory and manipulative skill ; you may know Wisden by heart, but if you can not

put your bat to the ball you will never be a cricketer. Ostrowski's method is to concentrate on a full discussion, taken at a fairly gentle pace but omitting no logical point of value, and to provide for the reader a large collection of exercises, well graded in each section, from the simple straightforward " rider " on the main theorems up to quite difficult results, theorems in their own right. Thus, for instance, the double integral is first defined over a rectangle (20 pages, 32 examples), we then return to the simple integral over a set, Leibnitz' rule for the derivative of an integral with respect to a parameter of the integrand, and the second mean value theorem (18 pages, 58 examples), and then we have the double integral over a general domain (15 pages, 34 examples, of which one is Hölder's inequality for double integrals). It must be noted that these 60 pages are mainly theoretical, for following them we have two further chapters, on the calculation of multiple integrals and on their applications. The same steady-going thoroughness copes with infinite integrals, simple and multiple, with such applications as a pleasing treatment of Frullani's integral, and the gamma function.

Now that the massive work is complete, we can assess the value of its three volumes as a whole. Of the modern books of this class, I would give Ostrowski's work pride of place, for its thoroughness, its splendid collection of interesting exercises, and its well-regulated speed of movement ; even at the most laborious pieces of argument we feel that the author is confidently progressing towards a goal which he has clearly envisaged. My one complaint would be that occasionally he is too formal ; the novice might well be able to follow each step and yet miss the crux of the argument. Apart from this comment, which may perhaps appear hyper-critical, I most heartily congratulate the author on bringing to fruition this solid and impressive task, and the publisher on the skill and craftsmanship which have gone to the making of a very fine specimen of modern book production. T. A. A. B.

Lehrbuch der Logistik. By K. Dürr. Pp. viii, 181. Sw. fr. 19.30 ; bound Sw. fr. 22.30. 1954. (Birkhäuser, Basel)

This elementary introduction to symbolic logic is by the Professor of Philosophy in the University of Zurich. It is written extremely clearly and exactly, illuminated by historical references, and maintains a reasonable balance between formal calculation and informal exposition ; both printing and production are first-rate.

The contents of the book include the propositional calculus, restricted and unrestricted predicate calculus, predicate calculus with equality, the descriptions operator, and the logic of classes and relations. The development of logic stops short of the introduction of numbers and there is no reference to metalogical concepts or theorems.

The propositional calculus is presented in the Polish, bracket-free, notation and there is an attractive device introduced which allows the truth table decision procedure to be applied in a simple natural manner. Negation, Np, is assigned the familiar valuations

p	Np
T	F
F	T

and disjunction Apq, conjunction Kpq and implication Cpq the valuations

p	q	Apq	Kpq	Cpq
T	T	T	T	T
T	F	T	F	F
F	T	T	F	T
F	F	F	F	T

The proof of the formula *CKpqApq* (that is, the conjunction of p, q implies their disjunction) runs as follows :

	C	*K*	*p*	*q*	*A*	*p*	*q*
1	*T*	*T*	*T*	*T*	*T*	*T*	*T*
2	*T*	*F*	*T*	*F*	*T*	*T*	*F*
3	*T*	*F*	*F*	*T*	*T*	*F*	*T*
4	*T*	*F*	*F*	*F*	*F*	*F*	*F*

In the third row, for instance, p and q are given the values F and T ; the value T for A is then found from the valuation table for A, and similarly the value F for K ; finally the value T for C is found from the table for C under the entries F, T for the arguments K and A. The calculation shows that C has the value T for all values of the variables p, q, and this constitutes the proof of the formula *CKpqApq*. R. L. GOODSTEIN.

Theory of functions of a complex variable, II. By C. CARATHEODORY. Translated by F. Steinhardt. Pp. 220. $4.50. 1954. (Chelsea Publishing Company, New York)

This translation of the second part of Caratheodory's last book is substantially a study of the mapping problem. As in the first part, the emphasis is on the geometrical aspect of function theory, and this perhaps is one reason why the book is easy to read. The first section, after some necessary preliminaries, deals with Riemann's key theorem on the conformal mapping of a bounded domain on a circle, the Koebe-Faber distortion theorem (*Verzerrungsatz*), and the mapping of the boundary. In the second section, the mapping of a curvilinear triangle bounded by three circular arcs is studied in detail, thus taking in the hypergeometric function, and this leads on to the famous theorems of Picard, Landau and Schottky. All this is basic stuff and to deal with it in 200 pages is no mean feat ; yet there is no sense of strain, and it is a measure of Caratheodory's mastery that he always appears to have plenty of elbow-room. Altogether a notable addition to the many existing books on function-theory, one which should be valuable to teachers and students alike. The Chelsea production is neat and pleasing, thanks mainly to a clear fount and generous spacing of formulae, and the translation, though stiff, is seldom obscure. T. A. A. B.

A treatise on the integral calculus, I, II. By J. EDWARDS. Rep. Pp. xxi, 907 ; xv, 980. $13. 1954. (Chelsea Publishing Company, New York)

In 1921–1922 this book horrified the professional analysts, and was rightly pilloried as an awful warning to the young. Today we are more likely to be mildly amused than shocked by the vague and cloudy language which only half-conceals the author's indefatigable capacity for blundering over even the most elementary logical points, and no one is going to erect Joseph Edwards into a pillar of sound doctrine. But we may still draw profit from his considerable manipulative dexterity, and from his vast store of curious information. Much of this information is not readily available elsewhere, and though a good deal of it is not likely to be wanted every day, we may well be glad to have a handy reference to topics such as Holditch's theorem and its extensions, the multiple integrals of Dirichlet and Liouville, and other half-forgotten lore.

The 2000 pages have been excellently reproduced by the Chelsea Company, and neatly bound. A period piece, without a doubt, but one for which many libraries and some private shelves may be glad to find room.

T. A. A. B.

Elements of algebra. By H. LEVI. Pp. 160. $3.25. 1954. (Chelsea Co., New York)

Professor Levi had the enviable task of giving a serious course on genuine mathematics to students whose main interests lay elsewhere; no previous knowledge of algebra was presupposed, but the students were expected to bring to their task an intelligent curiosity about the nature of mathematics.

Algebra is a particularly appropriate topic for such a study, since a considerable insight into structure can be acquired without utilising much complexity of manipulative technique, provided some taste for abstract argument is already present. The author takes the reader from sets to the construction of number fields, integers, rationals, real numbers, polynomials and simple equations, always with clarity and sometimes with a certain elegance in skirting thin ice.

For the intelligent non-mathematician who seriously asks "Why, ye gods, should two and two make four?" this little book might prove very stimulating and profitable. T. A. A. B.

Mathematics notes. Pp. 17. 1s. 1954. (Blackwell, Oxford)

Short notes for the G.C.E. at 0 level, to help the poorer pupil, who has covered the course with his teacher but needs extra assistance at certain standard obstructions. T. A. A. B.

The geometry of René Descartes. A facsimile of the 1637 edition, with a translation by D. E. Smith and M. L. Latham, reprinted. Pp. xiii, 244. Paper $1.50, cloth $2.95. 1954. (Dover Publications, New York)

The publication in 1637 of *La géométrie* was a mathematical event second in importance, in "the century of genius", only to the appearance of the *Principia*. For a full discussion of the content and import of Descartes' great work, we may refer our readers to the long review of the original edition of the Smith and Latham translation by Sir Thomas Heath, in the *Gazette*, XIII, pp. 258–263, while here contenting ourselves with a welcome for this neat and cheap reprint. T. A. A. B.

Real functions. By C. GOFFMAN. Pp. xii, 263. 30s. 1954. (Constable)

This book provides, in its way, an excellent introduction to the basic concepts and structures that underlie real analysis. The author has not intended it for the specialist, but as a textbook for the more mature student. The scope of the book is, therefore, deliberately restricted to what seems essential for a full understanding of modern analysis. The outstanding merit of the book is the really excellent selection of the topics, many of which are rarely found in similar texts, their skilful arrangement, and their elegant and reliable exposition. On the other hand, the scope of the book still remains so extensive that one doubts the wisdom of trying to compress it into a mere 250 pages. It is here that, in my opinion, the author has failed in writing a textbook that even the mature student can study with benefit on his own. The presentation is much too abstract, terse and condensed. No serious attempt has been made to relieve this icy atmosphere by some geometrical allusions or figures which would help the student to see the familiar pattern from which many of the concepts have been abstracted. What is more serious, many of the proofs even of important theorems (and not always simple proofs) have been left to the reader to supply, both in the text and in the numerous exercises. As for the latter, not even hints on how to attack them are given, which diminishes their usefulness greatly. It seems that the wish to condense as much as possible in a small volume was over-riding. What a

pity! A mere fifty pages or so, would have made all the difference. Nevertheless, the selection of the topics is so satisfactory and the proofs, so far as they are given, are so elegant that the book becomes a most welcome guide for the University teacher on which to model his own course in Analysis in a modern satisfactory way. Under his guidance, the student too will then benefit greatly from Professor Goffman's text. The following summary may give an idea of the rich content of the book. Real numbers, both Cantor sequences and Dedekind sections. Theory of sets, cardinals, ordinals, and the well-ordering theorem. Linear sets of points, notion of category. Metric spaces. Continuous and semi-continuous functions, their sets of discontinuity. Dini derivatives, derivation of monotone functions. Uniform convergence, at a point and on a set. Linear measure, Vitali's covering theorem. Borel sets and Baire classes. Riemann and Lebesgue integrals for one and two variables. Relation between differentiation and integration.

The book is beautifully printed, practically free from misprints, and reasonably priced.

W. W. ROGOSINSKI.

Reelle functionen. By G. AUMANN. Pp. viii, 416. DM 56; bound, DM 59.60. 1954. Grundlehren der mathematischen Wissenschaften 68. (Springer, Berlin)

Modern books on analysis are mainly concerned with the large structural features of the subject which they expose separately in their implications and in their mutual relationships, at the same time studying them in a more general abstract setting. The present book is a typical representative of this kind. It covers that trend of modern research which, in the wake of the classical treatises of Hahn, Carathéodory, and Hausdorff, deals with real functions and the theory of measure and integral. Functional analysis and the study of mappings, which is the second main branch of modern analysis, is hardly touched upon. The book is extremely well written, though in a severely formal style. Once the reader has become accustomed to this and the semi-logistic formalism employed, he will have no difficulty in studying it; and he will greatly benefit from its clear, competent, and reliable account. To the old-fashioned analyst of the classical school (the reviewer, for instance) the whole subject in its modern form may appear to have become rather sophisticated and lacking aesthetic appeal. This, however, is, no doubt, an understandable prejudice.

The following sketch can only indicate the general scope of the rich content. The book falls into three main parts. The introductory chapters dissect the real continuum into its ordering, algebraic and topological features which then are generalized to abstract order systems, to Boolean algebras of sets (somen algebras), and to general topological and metric spaces. Of the rich details we mention here Cantor's theory of cardinals and ordinals, and a general theory of convergence in an ordered system. The second part of the book studies continuous and semicontinuous functions on various topological spaces, deals with Baire classes arising from them, approximation theory (Stone-Weierstrass), iteration and fixed-points theorems, and ends with differentiation in metric linear spaces. The last part is dedicated to the modern theory of integration. It is mainly the F. Riesz approach to absolute integration which is studied in abstract form, on abstract spaces, and in several variations. The method of totalisation (Denjoy and Burkill), the Perron integral as a primitive function, and the theory of measure and its relation to the notion of the integral, are further topics to be mentioned.

The book is very well printed and practically free from misprints. A valuable list of the relevant literature is added.

W. W. ROGOSINSKI.

Mathieusche Funktionen und Sphäroidfunktionen mit Anwendungen auf physikalische und technische Probleme. By J. MEIXNER and F. W. SCHÄFKE. Pp. xii, 414. DM 49; bound, DM 52.60. 1954. Grundlehren der mathematischen Wissenschaften, 71. (Springer, Berlin)

Mathieu functions and spheroidal wave functions arise when the wave equation $(\nabla^2 + k^2)\phi = 0$ is separated in elliptic and spheroidal coordinates respectively, and are now becoming familiar to applied mathematicians. Several books dealing with these functions have already been published, e.g. in English, McLachlan's book (1947) on Mathieu functions, but the work now before us is the first one to give a mathematically rigorous account of their properties. The book is divided into four sections. The first section, of less than 100 pages, lays the mathematical foundations for the rest of the book, and although it is concise the presentation is clear. The second section of 120 pages gives an account of Mathieu functions, i.e. of the solutions of the equation

$$\frac{d^2y}{dx^2} + (\lambda - 2h^2 \cos 2x)y = 0$$

containing 2 parameters. The third section, of about 100 pages, gives an account of the spheroidal equation

$$(1 - z^2)\frac{d^2y}{dz^2} - 2z\frac{dy}{dz} + \left[\lambda + \gamma^2(1 - z^2) - \frac{\mu^2}{1 - z^2}\right] y = 0$$

containing 3 parameters. Mathieu's equation can be obtained from this equation by a transformation when $\mu = \frac{1}{2}$, but the treatment of Mathieu functions in the second section is not made dependent on that of the more general functions.

The last section, of 60 pages, gives an account of applications to problems involving elliptic or spheroidal boundaries in mechanics, electromagnetism, and quantum theory.

Both the authors have in the past made original contributions to the theory of these functions, and their book is a most welcome addition to the literature. Even if for everyday use some applied mathematicians will prefer the less condensed (if less rigorous) presentation of such standard works as McLachlan, they will be glad that a mathematical treatise is now available. Unfortunately few will be able to afford the high price (over £4). F. URSELL.

Existence Theorems for Ordinary Differential Equations. By F. J. MURRAY and K. S. MILLER. Pp. x, 154. $5. 1954. (New York University Press; distributed by Interscience)

It is not necessary to describe the contents of this book in detail, as the greater part of it has much in common with the chapters on existence theorems contained in well-known treatises on analysis such as Goursat's or the larger books on differential equations, such as Ince's. It will be sufficient to point out variations from the usual treatment.

The preface warns those who use electronic computers to solve differential equations that they need a proper mathematical basis, such as this study of existence theorems, to avoid wasting time and groping in the dark. The first existence theorem considered, based on Cauchy's use of Euler's polygonal lines, is not given as by Goursat or Ince, but in the improved form due to Peano and Arzelà, using a family of equicontinuous functions. The other two main existence theorems, Picard's method of successive approximations and Cauchy's calculus of limits (for analytical functions) are treated on the usual lines. In the chapter on linear differential equations use is made of *one-sided* Green's functions, matrices, and Jordan's theorem.

Throughout, the treatment is fuller than in the usual treatises. The authors are concerned not merely with giving a correct argument, but with pointing out to the reader exactly where the difficulties occur and how they have been overcome. H. T. H. PIAGGIO.

Vorlesungen über die Theorie der gewöhnlichen Differentialgleichungen. By I. G. PETROWSKI. Pp. 198. DM. 7.80. 1954. (Leipzig)

This is a German translation of the fourth Russian edition (1952), based on lectures delivered in Saratow as long ago as 1936–7. A previous edition received a Stalin prize of the second class in 1951. The greater part of the book consists of existence theorems. Like Murray and Miller, Petrowski gives the Peano–Arzelà form of the Cauchy–Euler polygonal line theorem, Picard's method of successive approximations, and Cauchy's calculus of limits. However, he also gives further theorems, sometimes in the text, as for Osgood's generalization of Lipschitz's condition for uniqueness, and sometimes in the examples. These examples are a most valuable feature of the book. They follow nearly every important theorem; many ask the student to construct proofs akin to those of the text when certain specified changes are made in the conditions. In the chapter on linear equations with constant coefficients the author uses elementary divisors, which are defined and explained for those without previous knowledge of them. They are also applied to determine the normal forms and hence the singular points of the differential equation

$$\frac{dy}{dx} = \frac{ax+by}{cx+dy}.$$

Although the title is restricted to *ordinary* differential equations, the book includes an appendix on partial differential equations, both linear and non-linear, of the first order. H. T. H. PIAGGIO.

Kugelfunktionen. By J. LENSE. 2nd edition. Pp. viii, 294. DM. 26. 1954. (Akademische Verlagsgesellschaft, Leipzig)

The last German standard work on spherical harmonics was Heine's treatise which appeared in 1878–81 and is now out of date. The present book is not intended as a full-scale treatise but rather to fill a gap as an advanced text-book in the German language for mathematicians, physicists, and engineers, and it is successful within this limited scope. The English-speaking public, however, will find most of the contents somewhere in Hobson's standard work. The book is divided into three sections, treating respectively spherical harmonics with integral index, spherical harmonics with arbitrary index, and applications to potential theory and applied mathematics. There are chapters on the gamma function and the hypergeometric function. The presentation is clear and attractive. The reviewer notes with regret that a modern standard treatise on spherical harmonics still remains to be written.

F. URSELL.

Entire Functions. By R. P. BOAS. Pp. xi, 276. $6. 1954. (Academic Press, New York)

This book deals with integral functions from a special point of view. Its aim is not unfairly described as the discussion of the validity of the classical interpolation expansions of Newton and Gauss, and their natural generalisations together with the immediate consequences of these expansions.

If we ask such questions as:
Does the fact that $f(n)$ is bounded for positive integer n imply that $f(x)$ is bounded for all positive x?

Does the condition $f^{(n)}(a_n)=0$ for a sequence a_n satisfying $|a_n| \leqslant 1$ imply that $f(x) \equiv 0$?

then this book provides the answers. It is more specialised than Whittaker's *Interpolatory function theory* for although not confined to integral function theory the excursions outside are incidental and the general theory of interpolation series is avoided.

The subject matter—apart from a 50 page introduction to the general theory of integral functions—is quite modern. There are very few references in the extensive bibliography outside the period 1940–1954. Nevertheless, the book is really an elementary one, the new ideas being notable for their originality of conception rather than their difficulty or depth.

The author has managed to arrange in a systematic way a very large number of individual contributions (many being his own) and in considerable detail at the end of each chapter, an analysis of the relevant sources.

A. J. M.

Geometrie der Zahlen. By O. H. KELLER. Enzyklopädie der Mathematischen Wissenschaften. Band 12, Heft 11, Teil III. Pp. 84. DM. 8.80. 1954. (Leipzig)

This article, which gives nearly 500 references to original papers, will undoubtedly be very useful to all who work in the subject and to anyone wishing to find what is known on a particular question. It is not likely to be useful to non-experts who may seek a general survey, as it is primarily a catalogue of results. The arrangement does not make it easy for a reader to concentrate his attention on the essentials of the subject, or to distinguish between the important and the unimportant. H. DAVENPORT.

Mathematics for the chemist. By G. J. KYNCH. Pp. vii, 356. 30s. 1955. (Academic Press, New York; Butterworth, London)

The author is Professor of Applied Mathematics at Aberystwyth and his purpose is to satisfy the needs of chemists, chemical engineers and physicists, whether students or those professionally qualified. In the reviewer's opinion, he has been more successful in providing a reference book for older practising chemists than a textbook for students taking degree courses. The latter should be able to dispense with some of the more elementary parts and ought to study the remainder in more detail and with more attention to mathematical technique acquired by means of examples. Many older chemists (not necessarily very old), on the other hand, took their degrees with little or no mathematics beyond A level of the G.C.E. and these will welcome a book which starts calculus again at the beginning, includes some revision of trigonometry and properties of curves and yet reaches such topics as multiple integrals, elements of vector field theory and the solution of the partial differential equation of diffusion, using Fourier series or Bessel functions. To cover so wide a range the author rightly gives many formulae without proof after careful explanation of fundamental principles.

The worked examples and those set for the reader include many applications of mathematics to chemistry and formulae drawn from atomic physics and thermodynamics. In all, however, there are only 166 exercises for the reader in twenty chapters and answers are not in general given. The research worker, referring to this book for help in expressing a specific problem mathematically, would no doubt be prepared to study the relevant section very closely and make further reference to books mentioned at the ends of chapters. Even so, he would be an able mathematician who, testing his comprehension of a chapter by means of examples, required no confirmation of his solutions. The largest set, of 28 questions, occurs at the end of the

chapter on differential equations and here the solutions of the equations of chemical kinetics are given with the questions. There is an interesting short section in this chapter on fast reactions, phenomena intermediate between ordinary chemical kinetics and nuclear fission. The book would not be out of place in the library of the sixth form at a grammar school, as an indication of the mathematics used in physical chemistry, or as a source book of examples of applied mathematics in the wider sense.

The only misprint noted is on p. 103, where a reference should be to qn. 7, not 8, of chap. 14. In chap. 3, the author sometimes uses the word " series " as the equivalent of " sequence " ; this is in conflict with his definition of series in chap. 13 and is particularly unconventional in the statement on p. 39 that " a series converges if every term lies between the two preceding terms, and the difference between successive terms tends to zero ".

C. G. P.

Mathematics of engineering systems (linear and non-linear). By D. F. LAWDEN. Pp. viii, 380. 30s. 1954. (Methuen)

This latest book on linear and non-linear systems differs from its predecessors in that it has been written by a mathematician instead of an engineer. This does not of course imply that it is necessarily a better or worse book than the others. Many of the latter have had authors with a considerable knowledge of mathematics who have presented their subject soundly and clearly. But the emphasis has very naturally been on practical considerations. Also engineers have a tendency to try to solve new problems by methods with which they themselves are familiar, and to think in terms of analogues or models rather than basic mathematics and to express themselves in the jargon of these related problems. There is therefore a place for a textbook which collects together the mathematical methods that are common to the theory of such subjects as mechanical and electrical vibrations, electrical circuits and their diverse applications, and automatic control systems. It can be stated at the outset that Mr. Lawden's book goes a long way to fill this need. Although, as he mentions in his preface, this is a text on mathematics, he shows with great care in worked examples how each of the topics introduced can be applied to specific practical problems. In other words he has faced up squarely to the task, which mathematicians teaching engineers sometimes tend to shirk, of showing how their subject ties up with the problems of the designer.

The first chapter consists of miscellaneous topics required later in the book. Some of these will certainly be revision and some will probably be new to the typical reader. They include functions, limits, expansion by Taylor's and Maclaurin's theorems and by long division, integrals and complex numbers. Although some of these subjects (such as integrals) have been dealt with in unnecessary detail for a book of this type one is conscious of certain important omissions, such as partial fractions. A commendable feature is a careful account of the methods of obtaining power series expansions, but again it is surprising to see no mention of the binomial theorem. The presence of this chapter means that the conscientious reader has to wade through fifty odd pages of preliminary matter with which he may be familiar before he gets down to the real subject of the book. For this reason it might have been better to put it in an appendix to be referred to if and when required. But coming first or last, it represents a most important part of the book. The teaching of engineers is so often hampered by inadequate knowledge and technique in the students of some of the more elementary branches of mathematics.

Since the commoner types of linear system are based on linear differential

equations with constant coefficients, these are discussed in considerable detail and with great care in the second chapter. The treatment follows the one that many of us will have met for the first time in Professor Piaggio's admirable textbook. Not only are there many numerical examples, but applications to circuits and simple servos are explained in great detail. This is a most useful feature, in spite of the fact that, applied extensively, it tends to increase the bulk of the book out of proportion to the ground covered. This chapter includes sections on simultaneous equations and on stability. It is somewhat surprising to see in the latter no mention of the name of Routh, who was not only an early investigator of this problem, but who produced a working rule which is as convenient to apply and a proof which is basically more direct than any of the more recent versions. True, the present book does contain the essence of Routh's approach, but in a more unwieldy form.

The third chapter is devoted to processes for finding the response to step and impulse functions and to sinusoidal functions, and the various methods used by engineers for representing and interpreting these solutions. These are based in the first place on the classical methods developed in the previous chapter, the Laplace transform being introduced at a later stage. A praiseworthy feature is a careful account of the behaviour of the system while a step or impulse in the input is being applied, although it is a pity that more stress has not been placed on the Laplace transform method here since, as is pointed out later, it is in this type of situation that it shows up to greatest advantage. A flaw in Mr. Lawden's approach has presumably led him to include the following exercise (p. 165). "Show that the application of unit impulse function as input to a system governed by the equation

$$(a_0D^n + a_1D^{n-1} + \ \ldots\ + a_n)\theta_0 = b_0D^n\theta_i$$

will evoke no transient response."

The important concepts of the Nyquist diagram and the weighting function receive due consideration in this chapter, which also includes a short section on differential-difference equations. The fourth chapter contains an account of Fourier series and integrals with applications, while the final chapter provides a very useful introduction to some of the standard methods for analysing non-linear systems. These include the phase plane and the various methods of perturbations, with particular reference to the equations of Duffing and van der Pol.

Although there are a number of minor mis-statements and other points open to criticism (for example, the function $e^{j\omega t}$ is described as a "cissoid") this book contains much valuable material. Mr. Lawden has taken great pains to make his explanations clear and the result is a very readable book. Every topic introduced is illustrated with carefully worked examples and augmented liberally by exercises for the student, for which answers are provided. In view of all this it is unfortunate that the general layout and typesetting leave much to be desired. In particular, it is not easy to see immediately where the sets of exercises interspersed in the text end and the narrative resumes. Also the system of numeration is such that cross references, which are many, are difficult to locate. For example, an attempt by the reviewer to find theorem 3.2 (iii) was abandoned since this appeared to require turning over anything up to eighty pages of chapter three. As regards typesetting, a number of the established conventions have been ignored. For instance, no attempt appears to have been made to avoid splitting formulae in the body of the text. However, even though these points may occasionally irritate or offend the aesthetic sense of the reader they do not detract from the great value of this book to the technical student or his teacher.

B. M. Brown.

Les fonctions de Bessel et leurs applications en physique. By G. GOURDET. 2nd edition. Pp. 90. 600 fr. 1954. (Masson, Paris)

This tract is directed mainly towards the physicist and engineer, but at this level it is naturally of interest to the mathematician concerned with the applications of his subject.

Bessel's equation is introduced by a consideration of the small vibrations of a circular membrane, and the usual solutions in series are developed. A long chapter follows on the mathematical properties of Bessel functions. This includes recurrence relations, Bessel functions of order $n+\frac{1}{2}$, generating functions, asymptotic values and Lommel integrals. The zeros of the Bessel function receive a bare mention. This chapter is so extremely well written that the author makes the reader feel that the whole topic is elementary. Another long chapter follows on applications of the previous results. Here one can browse on electromagnetic fields, skin effects in conductors, heat conduction, pressure waves, and diffraction, all presented with the clarity encountered previously.

There are some minor errors ; on page 6, for example, two mistakes cancel one another to give the correct result. However, these little blemishes will not deter the serious reader. A short bibliography is included, and there are abridged tables of the original and modified Bessel functions of both kinds, of orders zero and unity. M. HUTTON.

Minds and Machines. By W. SLUCKIN. Pp. 223. 2s. 1954. Pelican Book A308. (Penguin Books)

La Cybernétique. By P. COSSA. Pp. 98. 525 fr. 1955. (Masson, Paris)

The English book, in the Pelican series, contains a very readable study of the interactions between the ideas of cybernetics and the results of electronic engineering, neurology, physiology, biology, psychology—and metaphysics. The treatment is critical, though not unsympathetic. There is little of direct mathematical interest (except, perhaps, an elementary description of computing machinery), but much of interest to mathematicians.

By comparison, the French book reads like a popular science essay, with some stress on physiology. The style is pompous and descriptions of the "fancy that "-type are frequent. Titles of references are often misquoted. The last sentence reads : "It is paradoxical that this triumph of the human mind should induce some people, and even those who had some part in it, to deny the primacy of the Mind." S. VAJDA.

Probabilités, Erreurs. By É. BOREL, R. DELTHEIL et R. HURON. Pp. 220. 250 fr. 1954. (Colin, Paris)

This 9th edition of a well-known work on elementary theory of probability is said to have been revised and modernized, mainly by consistently using the concepts of random variables and of their distributions. It has lost nothing of its freshness and of the precision which is such a conspicuous and attractive feature of the best books in the French language. S. VAJDA.

Introduction to factor analysis. By B. FRUCHTER. Pp. xii, 280. 37s. 6d. 1954. (Van Nostrand, New York ; Macmillan, London)

Suppose that we have a mark sheet, giving the marks in t tests as obtained by p pupils. Then any mark can be regarded as being accounted for by the particular pupil's performance in respect of any common factor underlying all the tests, plus his performance on any factors that there may be in certain groups of tests, plus his performance on aspects specific to each test, plus his extra marks due to errors of measurement, these being due to peculiarities in the test, in himself, in the occasion, in the marker, and so on.

Can we find these common and group factors? There are more unknowns than there are equations, and any solution must state clearly its assumptions, its procedures, and its interpretations. This book, one of a Series in Psychology edited by Professor Guilford, is intended as " an introduction to the subject of factor analysis ". It reviews many of the procedures, though I do not think that it meets the demands that I have suggested above. There is a very full bibliography, of some 45 pages and 700 items, including a good representation of British workers. Some problems are set, to which answers are given, and there is a serviceable index. FRANK SANDON.

Factorial analysis for non-mathematicians. By C. J. ADCOCK. Pp. 88. 17s. 6d. 1954. (Melbourne University Press ; Cambridge University Press)

The writer of this book, the Senior Lecturer in Psychology at Victoria University College, Wellington, New Zealand, sets out " to cover the essential processes of factor analysis . . . to make clear the underlying logic in simple language ". I do not think that he succeeds. The logic does not reveal the assumptions, the approach of the build-up of two factors is not so clear as the similar approach of Vernon, the appearance of matrix algebra is inappropriate in a text for the non-mathematician, and the treatment of the " reality " of classifications depending on the choice of items in the test, or on the choice of subjects is unsatisfactory, and so on. The book therefore does not clarify the discipline at all satisfactorily. There is no index ; reference at the end is made to the books of Thomson and Burt and, with most emphasis, Thurstone.
FRANK SANDON.

Decision Processes. By R. M. THRALL, C. H. COOMBS, and R. L. DAVIES (editors). Pp. viii, 332. 40s. 1954. (Wiley, New York ; Chapman and Hall, London)

There have recently been many attempts, especially in America, to apply mathematical and statistical concepts to the sciences concerned with human social behaviour. These attempts have been inspired by such books as *The Theory of Games and Economic Behaviour* by von Neumann and Morgenstern, *The Mathematical Theory of Communication* by Shannon and Weaver, and *Statistical Decision Functions* by Abraham Wald. Such work calls for co-operation between all who can contribute to clarification and formulation of the problems to be solved, and so, in the kind of exploratory work reported in the book under review, we find mathematicians collaborating with philosophers, psychologists, sociologists and economists.

This exploratory work produces problems of different kinds for the mathematician. The pure mathematician is asked to solve technical problems and to provide new mathematical techniques ; the applied mathematician is needed to interpret the conclusions of mathematical theory and to abstract concepts amenable to mathematical treatment from psychological or sociological situations ; and the mathematical philosopher is invited to formulate axiomatic systems relevant to the new studies or to offer criticisms of the formulations offered by others.

The volume under review is of interest not for what it achieves, because that is admitted by the authors to be very little, but because it demonstrates clearly the mathematical probing of complex problems which have so far resisted systematic analysis. The nineteen papers reprinted here should stimulate every mathematical reader in some way, either to attempt to solve some of the problems described, or to elaborate some of the problems already solved, or to consider the relation between mathematical models and their physical interpretations. B. C. B.

Numbers in Colour. By G. CUISENAIRE and C. GATTEGNO. Pp. 39. 4s. 6d. 1954. (Heineman)

This little book takes its name from didactic material designed by G. Cuisinaire, Director of Education for Thuin, who in Part II describes the use of the apparatus for the teaching of reckoning in Belgian primary schools. The apparatus consists in the main of a large number of rods of lengths one to ten units. A single set of these thus forms the familiar "stair" but there is the added characteristic that equal rods have their own colour. Colours are assigned in families to draw attention to relationships between the numbers which the rod-lengths represent; for instance rod-lengths which are multiples of three units are distinguished by shades of blue and green.

The Preface contains three major claims for the method:

1. The material "can exemplify all the arithmetical relationships met with in school life." The foundations of this claim are outlined in Part I by C. Gattegno who writes with both primary and secondary British schools in mind.

2. The use of colour provides a "semi-abstract material" which has "overcome the obstacle of the gap between active and intellectual thought". This claim is clarified in the first chapter of Part II which ends with the significant statement that a bridge is formed between the experience gained through play and observation and the stage of systematic work. The book deals with one-way traffic on this bridge and throws light on the far side, abstract arithmetic. It has less to say of approaches to the bridge and the gap between real experience and the formal concrete material.

3. The method is "a *complete* answer to the mathematical and psychological problems set by the teaching of number as seen in arithmetic". Explanation of what is meant by this claim follows it, but its extreme form is unfortunate. It invites comment on the problems it does not include. Any method based on formal material, however adaptable, has inherent limitations and dangers. The material may not be needed as a bridge to abstract thought by all pupils or at all stages and as the book itself points out there can be danger in becoming too dependent on apparatus; it would be a pity if universally and permanently multiples of three were tinged a greenish blue. Moreover there are other bridges to be built if arithmetic is to be a usable accomplishment in everyday circumstances as well as a coherent intellectual system mastered in association with the rods. Again even if the apparatus seems to advance readiness for the intellectual stages of learning, it does not thereby solve this complex problem.

Teachers and intending teachers will find the emphasis on comprehension and the range of illustration in this book helpful even if they do not adopt the rods as an adjunct to their teaching. Non-specialist teachers of arithmetic will probably find reading easier if they begin with Part II, ignoring the age allocations which belong to the Belgian syllabus. L. D. A.

Algebra. Its big Ideas and Basic Skills. Book 1. By D. J. AIKEN and K. B. HENDERSON. Pp. xi, 419. 22s. 1954. (McGraw-Hill)

I found this very readable book quite fascinating. Its freshness of approach is such that any 11-year-old who picked it up by accident might be led to go on reading it almost for pleasure. He would delight in the "comic-strip" cartoons, each of which teaches a valuable lesson, and he might even want to solve the "merry-go-round puzzles", such as: 1. $2a - 16 = 5(a - 5)$, 2. $5 + 10b = 2b + 45$, 3. $a^a + b + b^a = y$, 4. $x^2 - 2xy + y^2 = 0$, the answers to which are cleverly concealed in pictures shewing, for instance, three imps mounted on eagles chasing each other round a kangaroo.

The seven "big ideas" are: General Numbers, Equations, Signed Numbers, Dependence, Graphical Representation, Exponents at Work and

Indirect Measurement, which is an introduction to trigonometry. The painstaking explanations are designed not only to shew the pupil what he should do, but also what he should not and why : they should arouse the interest of the average boy, and make him see reason in what he is learning, though perhaps the very best might grow a little tired of the constant driving home of principles, and feel that the explanations in " words of one syllable " are beneath him. But this is a good fault, and the continual reminders of how to learn (" Do not ask for help until you have put forth an honest effort : work until you can put your finger on what you do not understand, then you can ask for help intelligently ") are not the least valuable part of the book. Each of the first twelve chapters ends with a " Summary of things to remember " in three paragraphs, 1. The Key Idea ; 2. Important Words and Expressions ; 3. Important Skills. There are plenty of quite simple and well-chosen exercises, Chapter Tests and Valuable Skills Practices. The final chapter contains a review of the previous work and some supplementary topics, e.g. solution of quadratics by formula.

Though the book gives an excellent introduction to algebra it would be difficult to fit it into an English school course, since it does not cover even *O* level work : there is, for example, no real work on logarithms, though the subject is slightly introduced under exponents, and there are no log. tables. Weights and Measures, Square Roots and elementary Trigonometrical Ratios are given in an Appendix. There is little to criticise : I do not like the statement " if $(x-5)(x-6)=0$ then either $x-5=0$ or $x-6=0$, or both may equal zero ", an exercise is repeated on page 277, and the solutions on page 321 have gone wrong. As usual in American books, this is well printed on excellent paper. B. A. S.

Algebra. By E. H. Lockwood and D. K. Down. Pp. vi, 438. 11s. 1954. (University of London. London Mathematical Series)

Written by experienced teachers at Felsted and Loughton High Schools, this book provides an introduction to Algebra for 11-year-olds and covers the usual G.C.E. " O " level work. The subject is developed from the equation, the formula coming in chapter 3, and an unusual feature in modern textbooks is the chapter on " The Rules of Algebra ", in which " the vice-versa rule ", " the any order rule " and " the do-it-to-each rule " are discussed in a quite attractive manner. There is practically no bookwork—each topic is introduced by means of a " teaching exercise ", and the authors believe that the ablest pupils will acquire the ideas with little, if any, help from the teacher. I am a little doubtful about this claim, because though the teaching exercises are generally comprehensive, there are gaps occasionally which can only be filled in by the teacher. I wonder, too, whether the pupil may not be overwhelmed by the number of exercises ; there are 21 chapters, of which the first five contain 1026 exercises, and in addition there are 48 Revision Papers. There is no teaching on the manipulation of negative characteristics or the setting out of logarithms. The book ends with a bare introduction to the Binomial through Pascal's Triangle, and the essential ideas of the calculus by means of finite and small differences, and the summation method for areas ; the teacher's help will certainly be necessary here in order to prevent misconceptions about limits. There are tables of logs., antilogs. and square roots—and 102 pages of answers. B. A. S.

Mathematical Puzzles and Pastimes. By Aaron Bakst. Pp. vi, 206. 27s. 1954. (Macmillan)

This necessarily somewhat formless book is intended to be, not a collection of mathematical puzzles, but an exposition of the principles underlying certain

mathematical curiosities from which puzzles can be constructed. Nevertheless it begins with a collection of 36 problems of the type " Remove two toothpicks from figure 2 leaving two squares ", though Dr. Bakst admits that there are no general principles in matchstick geometry. The next chapter is the most original in the book. Using a perfectly elastic billiard ball on a table in the form of a 60° parallelogram with one corner cut off, Dr. Bakst shews how to solve, or prove insoluble, the problem of three containers of different capacities, of which the largest is full of a liquid which is to be distributed among the containers in stated proportions. There are four chapters on scales of notation which have some points of interest; they mention one or two well-known puzzles, such as " Russian peasant multiplication " and weighing with four weights, but do not seem to lead far except for a brief statement of the use of the binary and bi-quinary scales in electronic computers. A chapter on tests for divisibility is followed by one on time which introduces the perpetual calendar and Easter date calendar. Some interesting unicursal figures are followed by an undistinguished chapter on π: rolling circles and the path of an aeroplane travelling along a meridian as viewed from outer space: the construction of tables of sines, cosines and tangents for small angles and then by the use of the addition formulae (none of these rules is proved), and a long and sketchy proof of a formula for finding the distance between two points on the earth's surface: arithmetic by geometrical construction: and finally some miscellaneous number problems such as " $ab = a + b$ " and " find a number which can be divided by 3 by transferring the left-hand digit to the right ".

The book shews signs of careless writing and proof reading; there are far too many errors and ambiguities which cannot be accounted to the printer. I have found mistakes on pages 41, 49, 63 (the soroban figure), 76, 92 (a bad mistake; both the rule and the arithmetic for finding the day on which a given date falls are wrong), 140, 163 and 175, and ambiguities or obscure writing on pages 44, 99, 145–8, 155–6: the length of the year assumed on page 90 disagrees with that given on p. 86. The price is heavy for English pockets.

B. A. S.

Arithmetic made Easy. By W. HAYDN RICHARDS. Book 4, Part 1, Pp. 119; Part 2, Pp. 104. 3s. 6d. each. 1955. (Harrap)

These two small books complete the junior school course. The author has tried to think of every possible difficulty in calculation, and to make a " rule " to meet it, but his method depends entirely on giving sufficient practice to fix the rule in the memory, and makes no appeal to a pupil's reason. The course contains all the traditional arithmetic of the primary school—the suggested ways of setting out written work are modern, and on the whole the numbers involved are not too large. There is considerable ingenuity shown in giving variety to the drill exercises, and in constructing small problems to illustrate each rule. In part 1, page 11, the plus signs are omitted, and in part 2, page 22, the word " multiplication " is wrongly used. Also, in part 1, page 88, " sharing " is used as as technical term—surely the answer to a " sharing sum " should be a share, not a quotient!

H. M. C.

Mathematics for Living. By E. R. HAMILTON and C. H. J. SMITH. Book 4: Living in a community. Pp. 192, including 9 pages of tables. Limp, 5s.; boards, 6s. Teacher's edition: pp. iii–vii, introduction, a reprint of that in Book 1; 192 as in pupil's book, 193–211 numerical answers. Limp, 6s. 6d.; boards, 7s. 1954. (University of London Press)

Like the other books of the series this last one is intended to be used with a teacher, and one who is keen will find it very suggestive. There is a greater

emphasis in it on mathematics as such, and about sixty pages, not consecutive, are devoted directly to the learning of logarithmic calculation, the study of angles, triangles, and circles, geometrical drawing, and various kinds of graphs. Nearly as many pages contain the applications to citizenship from which the book takes its title, and the chapter headings here are insurance, hospital and health services, parks and playing fields, water-supply, gas-supply, sewage disposal, births and deaths, finances of a borough, road-making, road accidents, income tax, and the budget. Perhaps surprisingly, it is in the chapter on gas-supply that the opportunity is made to teach the use of tangent tables, and also the word " parabola ". The tables given also include sines and cosines, but no similar opportunity of introducing them appears to have been made. A great deal of practice in mensuration has been contrived by interesting pupils in the many miles of pipes of varying sizes which carry gas and water and sewage under our streets. The reading of tables and graphs of statistics in considerable variety has also received attention.

Some of the examples, however, e.g. the statistics about diphtheria and tuberculosis, and the suggestions about the size of playing fields in relation to the number of pupils in a school, would seem more suitable for training college students and adult classes than for boys and girls of fifteen, and it is unfortunate that on page 38 it is implied that it takes the whole of the national insurance contributions to provide the health services and sickness benefits whereas they also cover unemployment insurance and industrial pensions. The examples in exercise 81 where the salary has to be deduced from the income tax paid, instead of the other way round, seem somewhat unpractical. Two more mathematical criticisms are : (*a*) fig. 5 on page 11 is on a different scale from fig. 4, and several numbers which are not " tens " have been inserted, so the heading is misleading ; (*b*) in exercises 67 and 68 there is no indication of what conventional projection is to be used for the scale " pictures ", nor any suggestion that the teacher should be consulted.

At intervals throughout the book, and occupying some sixty-three pages in all, are sets of miscellaneous exercises under the heading " revision ". Like other exercises these are marked E (easiest), A or H, but whether a pupil working steadily through all the E exercises might be expected to find them gradually becoming more difficult is not stated. It is likely that to many a pupil each problem would appear new, in spite of occasional references back —if he really wished to revise he would want a group all on one subject.

H. M. C.

Advanced Level Pure Mathematics, III. By S. L. Green. Pp. xi, 285–624. 12s. 1954. (University Tutorial Press)

Advanced Level Pure Mathematics by S. L. Green is now completed by the third part, dealing with Algebra and Analysis. The bookwork is presented in an admirably clear manner, and a feature is the large number of exercises worked in the text. The examples are of the type encountered in Advanced and Scholarship level papers. Differential and Integral Calculus are treated ab initio, although the author, quite properly, assumes that the groundwork has been previously covered. It is a pity that space could not be found for an introduction to partial differentiation, which is one of the requirements at this stage. Complex numbers and determinants are introduced in the last two chapters.

The book is attractively printed, the examples sufficiently numerous, and misprints rare (e.g. two missing δ's in paragraph 282) and the author is to be congratulated on the amount of important material he has compressed into its pages.

W. J. Hodgetts.

Variational Principles in Dynamics and Quantum Theory. By W. YOURGRAU and S. MANDELSTAM. Pp. ix, 155. 25s. 1955. (Pitman)

This little book, inspired by Planck's interest in the principle of least action, gives an account of variational principles in physics. The first half of the book is devoted to classical mechanics. The authors confine their attention to the principle of least action and the principle of Hamilton, and these principles they discuss only for holonomic systems. They next consider variational principles in electrodynamics and in quantum theory, and the book ends with a chapter containing an assessment of the significance of variational principles in the general structure of scientific thought. The authors have chosen to discuss the subject in the historic order of its development, and they describe the contributions to the subject of Fermat, Huygens, Maupertuis, Euler, Lagrange, and Hamilton. Some of the original proofs are reproduced, and these quotations from the pioneer workers in this field form an interesting and valuable feature of the book. But the emphasis is not entirely happy : the unwary reader might come away with the impression that Maupertuis was a figure of nearly the same stature as the great Euler himself!

Hamilton's principle is, in my view, a simpler principle than the principle of least action, because Hamilton's principle involves a straightforward variation problem, whereas the principle of least action involves a " Lagrange problem " in which attention is restricted to curves satisfying a subsidiary differential equation. Correspondingly the principal function is a simpler function than the characteristic function. For example, in the trivial problem of a particle moving in a uniform field, the terminal points and the *time of transit* determine a unique path, so the principal function is uniform : but the terminal points and the *energy constant* give two paths in general, as every schoolboy knows, so the characteristic function is multiform. This simple illustration should have saved the authors from the unfortunate remark (p. 50) that the terminal points and the energy are " sufficient to determine uniquely the path and hence the action "—though fortunately the mistake is corrected in a footnote. From the point of view of a reader to whom the subject is new the historical approach has the disadvantage that the principle of least action comes before Hamilton's principle, and the characteristic function occurs before the principal function. I think an exposition in the reverse order would be easier to understand and would give a deeper insight into the subject.

Some of the explanations seem somewhat obscure. For example, the Lagrangian function for an electron (of variable mass) is found on p. 39, and the formula is correct, but the argument is not easy to follow. The account of quasi-periodic motions in § 10 seems to me altogether too condensed to be intelligible to anyone who has not met the theory before. There are a few unfortunate misprints, for example the omission of the indices in the Gauss constraint function (p. 96), and of the coefficients in the multiple Fourier series on p. 106. The last chapter recalls some well-worn controversies, and is not altogether easy reading.

L. A. P.

CORRECTION TO NOTE 2539.

The diagrams in this note have unfortunately become confused in printing. Figs. 2 and 3 should be interchanged, but not the legends attached to them; and in Fig. 4 the central pentagon of which S is a vertex should be hatched, and the whole of the pentagon of which CQ is an edge should be stippled — i.e. the stippling should be continued to include the triangle to its left.

H.M.C.

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